Advanced Calculus

Rose-Hulman Institute of Technology

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R.J. Marks II Class Notes

(1970)

CALC

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GOD BLESS HIS SOUL!

9-10-70

MORE OF SAME THEORY PROOF'S OF THINGS YOU'VE ALREADY SEEN

1) INDUCTIVE SET - A SET OF REAL NUMBERS
2) $\ni \Rightarrow$ SUCH THAT
3) $\Rightarrow \Rightarrow$ IMPLIES
4) $\in \Rightarrow$ IS AN ELEMENT OF
5) OBSERVATION - THE POSITIVE INTEGERS $\subseteq S$ WHERE S IS ANY INDUCTIVE SET
6) $\forall \Rightarrow$ FOR EVERY

CONSIDER: $Q = \cap \{S \mid S$ IN INDUCTIVE SET$\}$
$(x \in Q \Rightarrow)\ x \in S$
PAST INT. $\subseteq S \Rightarrow$ POS. INT. $\subseteq \Omega \{S/S$ IS IND. SET$\}$
POST INT $\subseteq \{S/S$ IS INDUCTIVE$\}$
$\therefore Q$ IS POSITIVE INTEGER

DEFINITION: LET $PI = \{n \mid n$ POS INT.$\}$
THEN $PI = \cap \{S/S$ IS AN INDUCTIVE SET$\}$
PI, THE POSITIVE INTEGER IS THE SMALLEST INDUCTIVE SET

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$\Leftarrow$ IS IMPLIED BY

$\Leftrightarrow$ (=) IMPLIED AN IMPLIED BY

$\subseteq$ IS A SUBSET OF

$\exists$ THERE EXISTS

$\exists|$ THERE EXISTS UNIQUELY

AN INDUCTIVE SET, A SET OF REAL #'s

$1 \in S$; $x \in S \Rightarrow x+1 \in S$

WE PROVED THAT THE SMALLEST INDUCTIVE SET P.I. OF POSITIVE INT.

<u>AXIOM OF INDUCTION</u>

LET S BE A SET OF POSITIVE INTEGERS, WHICH IS INDUCTIVE THEN $S \Rightarrow$ P.I.

PROOF: S A SET OF POSITIVE INTEGERS $\therefore S \subseteq$ P.I.

S INDUCTIVE P.I. $\subseteq S \therefore S =$ P.I.

PRINCIPLE OF MATHEMATICAL INDUCTION

LET $P(n)$ BE A PROPOSITION WHICH DEPENDS ON THE INTEGER n, & WHICH CAN BE EITHER TRUE OR FALSE

IF $P(1)$ IS TRUE, $P(n)$ TRUE, $\therefore P(n+1)$ IS TRUE

THEN $P(n)$ IS TRUE FOR ALL n

<u>PROOF</u>

LET $S = \{ n / P(n) \text{ TRUE} \}$

$S \subseteq P.I. \quad 1 \in S \Rightarrow n+1 \in S$

$\therefore S$ IS AN INDUCTIVE SET OF POSITIVE INTEGERS

EXAMPE OF APLIED MATH:

$$P(n) : \sum_{K=1}^{n} K = n(n+1)/2$$

PROOF OF BINOMIAL THEOROM

$$(X+Y)^N = \sum_{K=0}^{n} \binom{n}{K} X^n Y^{n-K}$$

$$= n! / K!(n-K)!$$

$$P(1) = (X+Y) = \sum_{0}^{1} \binom{1}{K} X^K Y^{1-K}$$

$$= X^0 Y^1 + X^1 Y^0 = X + Y$$

ASSUME $(X+Y)^n = \sum_{K=0}^{n} \binom{n}{K} X^K Y^{N-K}$

$$(X+Y)^{n+1} = (X+Y)^n (X+Y)$$

$$= X(X+Y)^n + Y(X+Y)^n$$

$$= X \sum_{K=0}^{n} \binom{n}{K} X^K Y^{n-K}$$

$$+ Y \sum_{K=0}^{n} \binom{n}{K} X^K Y^{n-K}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k}$$

$$+ \sum_{k=0}^{n} \binom{n}{k} \dot{x}^{k} y^{n+1-k}$$

(ARG!)

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$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$\binom{n}{k} \stackrel{\text{DEFN}}{=} \frac{n!}{k!(n-k)!}$$

(n CHOOSE k)

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \binom{n}{0} y^n + \binom{n}{1} x y^{n-1} + \binom{n}{2} x^2 y^{n-2} \ldots$$

$$\binom{n}{0} = 1 ; \quad \binom{n}{1} = n ; \quad \binom{n}{2} = n(n-1)$$

LOOK AT $P(n+1)$

[redacted]

$$(x+y)^{n+1} = (x+y)(x+y)^n = x(x+y)^n + y(x+y)^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k+1}$$

$$j = k+1$$

$$P = \sum_{j=1}^{n+1} \binom{n}{j-1} x^j y^{n-j+1} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k+1}$$

$$= \overset{=1}{\binom{n}{0}} y^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k} \right] x^k y^{n-k+1} + \overset{=1}{\binom{n}{n}} x^{n+1}$$

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!\,(n-k+1)} + \frac{n!}{k!\,(n-k)!}$$

$$= \frac{\overset{(n+1)!}{\cancel{n!\,k + n!\,(n-k+1)}}}{k!\,(n-k+1)!}$$

$$= \frac{(n+1)!}{k!\,(n+1-k)!} = \binom{n+1}{k}$$

WELL ORDERING OF POSITIVE INTEGERS

THRM: LET N BE A NON-EMPTY SET OF POS. INTEGERS. N HAS A SMALLEST ELEMENT

$N \subseteq PI,\ N \neq \emptyset,\ \exists\, y \in N \ni y \leq x\ \forall x \in N$

PROOF: ASSUME N HAS NO SMALLEST ELEMENT

LET $B = \{ z \mid z \in PI,\ x > z\ \forall x \in N \}$

$1 \in B$; $1 \leq$ ANY POS. INT., MOREOVER $1 < x\ \forall\, x \in N$ FOR WE ASSUME N HAS NO SMALLEST ELEMENT

LET $n \in B$ $\quad n < x\ \forall x \in N$

$\therefore\ n+1 \leq x\ \forall x \in N$

BUT $n+1 \neq x$ FOR ANY $x \in N$

$\therefore\ n \in B \Rightarrow n+1 \in B$

$\therefore\ B = PI$

$\therefore\ N = \emptyset$, WHICH IS IN COUNTERDICTION TO INITIAL ASSUMPTIONS

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AXIOM: AXIOM OF COMPLETENESS:
ANY SET OF REAL NUMBERS WHICH IS BOUNDED ABOVE; HAS A LEAST (IE SMALLEST) UPPER BOUND.

IE) LET $A \subseteq R$; M IS AN UPPER BOUND FOR A IF $x \leq M \; \forall x \in A$

EX) $A = \{ 1 + \frac{(-1)^n}{n} / n \in P.I \} = \{ 0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4} \dots \}$
2 IS AN UPPER BOUND

EX) $B = \{ (1+n)^{1/n} / n \in P.I. \}$ 3 IS AN UPPER BOUND

$$\lim_{n \to \infty} (1+n)^{1/n} = l$$

DEFINITION: IF A LEAST UPPER BOUND FOR A SET A, IS AN UPPER BOUND L, $\ni$ IF M IS ALSO AN UPPER BOUND, THEN $L \leq M$

L.U.B. = LEAST UPPER BOUND
SUB = SUPREMUM (L.U.B.)

$A = \{ (1 + \frac{1}{n})^n / n \in P.I \} \subseteq$ RATIONAL #
THERE IS NO NUMBER IN A WHICH IS A LEAST UPPER BOUND FOR A

⇒ ARCHEMEDIAN PROPERTY OF THE POSITIVE REAL NUMBERS

THEOREM: LET a, b BE REAL POSITIVE NUMBERS.

THEN $\exists,\ n \in P.I \ni na > b$

PROOF: ASSUME $na \leq b \ \forall n \in P.I.$

LET $S = \{ na \mid n \in PI \} \subseteq R$

b IS AN UPPER BOUND FOR S

$\therefore$ S HAS A L.U.B., CALLED c

$\therefore$ $na \leq c \ \forall n$ IN PARTICULAR

$(n+1)a \leq c$

$\therefore$ $na \leq c - a < c$

COR. 1: IF x IS ANY REAL #, $\exists$

$\exists n \in PI \ni x < n$

PROOF: IF $x \leq 0$, LET $n = 1$

IF $x > 0$ LET $a = 1$; $b = x$

$n > x$

COR. 2: IF $\epsilon > 0$ IS ANY REAL # $\exists$

$\exists n \in PI \ni \frac{1}{n} < \epsilon$

~~COR/2.~~ PROOF:

LET $x = \frac{1}{\epsilon}$ BY (1) $\exists n \ni x < n$

i.e. $\frac{1}{\epsilon} < n$: $\frac{1}{n} < \epsilon$ Q.E.D.

COR. 3: IF ~~XXX~~ x IS ANY REAL # $\exists\ m, n \in I \ni m < x < n$

PROOF:

BY (1) $\exists\ n \ni x < n$

BY (1) $\exists\ p \ni -x < p$

$\therefore -p < x < n$

THEM. II: IF $x \in R\ \exists!$ ($\exists$ A UNIQUE) $n \in I \ni n \leq x < n+1$

PROOF: BY COR (3) $\exists\ r, s \in I \ni r < x < s$

LET $A = \{n / n \in PI \text{ AND } n + r > x\}$

$A \neq \emptyset$ BECAUSE $s - r \in A$.

NOW $A \subseteq PI$ BY WELL ORDERING OF PI, $\exists\ p$, A SMALLEST POS INT $\ni r + p > x$

LET $n = r + p - 1$

CLAIM

$n \leq x < n+1$

TRUE BY DEF. OF n

IF $p = 1$; $r \leq x < r+1$

IF $p \geq 2$:

$n = r + p - 1 \leq x$ WHY?

$\because p$ IS THE SMALLEST # FOR WHICH $r + p > x$

UNIQUENESS:

ASSUME $\exists$ $m, n, m<n$

$\ni m \le x < m+1$

$n \le x < n+1$

$m < n \le x < n+1$

CONTRODITE V ON PG. 8

IF THERE IS A POS INTEGER BETWEEN n AND $n+1$

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THM: THE RATIONALS ARE DENSE IN THE REAL #'s.

ie) $a, b \in R$, $a<b$, $\exists$ A NATURAL # $r \ni a<r<b$. MOREOVER $\exists$ AN INFINITE # OF SUCH RATIONALS

PROOF:

$b-a>0$

$\therefore \exists\ q \in PI \ni \frac{1}{q} < b-a$ (COR III)

BY THM. II, $\exists p \in I \ni p \le aq+1 < p+1$

$\therefore aq < p$ or $a < p/q$

$\frac{p}{q} \le a + \frac{1}{q} < b$

$\Rightarrow a < p/q < b$ Q.E.D.

MATH INDUCTION; L.U.B, RATIONAL DENSITY

KNOW FROM CHAPT. 1

COMPLETION OF THE RATIONAL #'S

MAY USE 3 METHODS

(1) L.U.B.

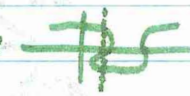
(2) DEDEKIND CUT POST.

(3) CONV. OF CAUCHY SEQ. OF RAT. #'S

CHAPT. 2; FUNCTIONS & SEQUENCES

ASSUME: YOU KNOW WHAT A FUNC. IS DOMAIN, RANGE, INDEPENDENT AND DEPENDENT VARIABLES

$\forall x \in D \ \exists! \ y \in R$

FCT $\Rightarrow$ SINGLE VALUED:  NOT 

DEFINITION: A SEQUENCE IS A FUNCTION f, WHOSE DOMAIN IS THE POS. INT, & WHOSE RANGE IS CONTAINED IN THE REAL

ie) $f = \{(1,a_1),(2,a_2),(3,a_3)\ldots\}$

a_k ARE THE TERMS OF THE SEQ.

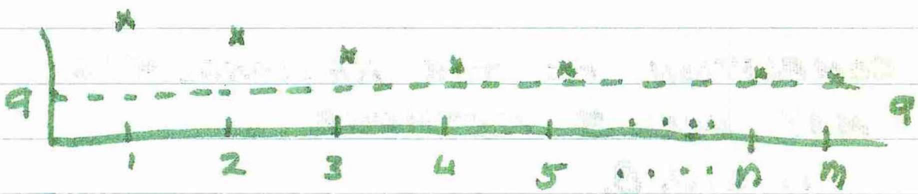
& WE USUALLY ABBREVIATE THE INDICATION OF A SEQ AS FOLLOWS:

$a_1, a_2, a_3 \ldots a_n \ldots$ OR $\{a_n\}$

EX) $1, 2, 1, 3, 1, 4, 1, 5 \ldots$

$a_{2n-1} = 1$

$a_{2n} = n+1$



DEFINE: LET a_n BE A SEQ. WE SAY THAT THE SEQ HAS LIMIT A, IF

$\forall \epsilon > 0, \exists N(\epsilon) \in PI, \ni n > N(\epsilon)$

$\Rightarrow |a_n - A| < \epsilon$

& WE WRITE $\lim_{n \to \infty} a_n = A$

EX.) PROVE $\lim_{n\to\infty} \frac{1}{n} = 0$

LET $\epsilon > 0$, $|\frac{1}{n} - 0| < \epsilon$

$\Leftrightarrow |\frac{1}{n}| < \epsilon$, BUT $n > 0$

$\Leftrightarrow n > \frac{1}{\epsilon}$ $\therefore N(\epsilon) = n$ IS SMALLEST INTEGER LARGER THAN $\frac{1}{\epsilon}$

EX) T.B.P. $\lim_{n\to\infty} \frac{n}{n+1} = 1$

LET $\epsilon > 0$ $\left|\frac{n}{n+1} - 1\right| < \epsilon$

IFF $\frac{n-(n+1)}{n+1} < \epsilon \Leftrightarrow \frac{1}{n+1} < \epsilon$

$\Leftrightarrow n+1 > 1/\epsilon$

$\therefore N(\epsilon) =$ SMALLEST INTEGER LARGER THAN $\frac{1}{\epsilon} - 1$

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DEFN: WE SAY THAT

$$\lim_{n\to\infty} a_n = a, \text{ IF } \forall \varepsilon > 0$$

$$\exists\ N(\varepsilon) \in PI \ni n > N(\varepsilon) \Rightarrow |a_n - a| < \varepsilon$$

1) EX) PROVE $\lim_{n\to\infty} \frac{n}{n+1} = 1$

LET $\varepsilon > 0$, THEN $\left|\frac{n}{n+1} - 1\right|$

$$= \left|\frac{n-(n+1)}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \varepsilon$$

IF $\frac{1}{\varepsilon} < n+1$ OR $\left[n > \frac{1}{\varepsilon} - 1\right]$

LET $N(\varepsilon) = \begin{cases} \text{~~LARGEST~~ SMALLEST INTEGER LARGER THAN } 1/\varepsilon - 1 \\ \text{PROVIDED } (1/\varepsilon - 1) > 0 \\ \text{OTHERWISE, } N = 1 \end{cases}$

2) EX) $\lim_{n\to\infty} e^{-n} = 0$

TO PROOF: LET ~~e^{-n}~~ $\varepsilon > 0$ BE GIVEN. $|e^{-n} - 0| = e^{-n} < \varepsilon$ IF

$$\ln e^{-n} < \ln(\varepsilon)$$

$$\Rightarrow +n > -\ln(\varepsilon)$$

SO LET $N(\varepsilon) = \begin{cases} \text{SMALLEST INT} \geq -\ln\varepsilon \\ \text{IF } \varepsilon < 1, 1 \\ \varepsilon > 1. \end{cases}$

(CONT.)

3) EX) $\lim_{n\to\infty} \frac{e^{-n}}{n} = 0$

PROOF: LET $\varepsilon > 0$

$$\left| \frac{e^{-n}}{n} - \overset{\text{PROPOSED LIMIT}}{0} \right| = e^{-n}/n < e^{-n} < \varepsilon$$

$\therefore$ CHOOSE ε AS IN THE PREVIOUS EXAMPLE

4) EX) $\lim_{n\to\infty} \frac{n^2-1}{n^2+1} = 1$

PROOF: LET $\varepsilon > 0$

$$\left| \frac{n^2-1}{n^2+1} - 1 \right| = \left| \frac{n^2-1-(n^2+1)}{n^2+1} \right|$$

$$= \frac{2}{n^2+1} < \frac{2}{n^2} < \varepsilon$$

$\therefore n > \sqrt{2}/\sqrt{\varepsilon}$

LET $N(\varepsilon)$ = ANY INT LARGER THAN $\sqrt{2/\varepsilon}$

THEM:

IF $\lim_{n\to\infty} a^n$ EXISTS, THEN IT IS UNIQUE.

ASSUME a & b ARE DISTINCT REAL #'S $\ni \lim_{n\to\infty} a_n = a$

AND $\lim_{n\to\infty} a_n = b$

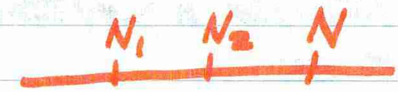
(CONT.)

(CONT)

LET $\varepsilon > 0$

$\exists N_1 \ni n > N_1 \Rightarrow |a_n - a| < \varepsilon$

$\exists N_2 \ni n > N_2 \Rightarrow |a_n - b| < \varepsilon$

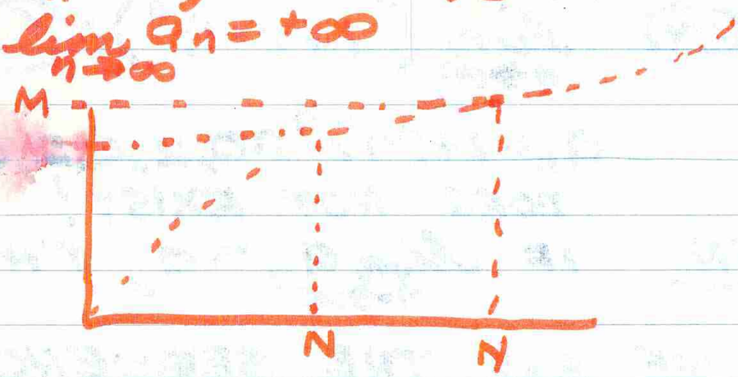


LET $N = \text{MAX}\ \{N_1, N_2\}$

$\therefore n > N \quad |a_n - a| < \varepsilon \ \& \ |a_n - b| < \varepsilon$

~~WITH~~ $\varepsilon = b - a/3$ THIS IS IMPOSSIBLE, $\therefore$ THE CONCLUSION IS FALSE, ERGO, THE LIMIT IS UNIQUE

DEFN: IF $\forall M > 0 \ \exists N(M) \ni n > N \Rightarrow a_n > M$, THEN WE SAY $\lim_{n \to \infty} a_n = +\infty$



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DEFN.

$\lim\limits_{n\to\infty} a_n = \pm\infty$, IF $\forall M>0$ (HOWEVER LARGE) $\exists N(M) \ni n>N \Rightarrow a_n > M$

DEFN $\lim\limits_{n\to\infty} = \infty$

IFF $\lim\limits_{n\to\infty} |a_n| = +\infty$

EX $a_n = (-2)^n = (-1)^n(2^n)$
$= -2, 4, -8, 16, -32, 64 \ldots$

$\lim\limits_{n\to\infty} -2^n = \infty,$

$\therefore \lim\limits_{n\to\infty} |(-2)^n| = \lim\limits_{n\to\infty} 2^n = \infty$

EX $\lim\limits_{n\to\infty} (1+(-1)^n) = \boxed{}$

$a_n = 1+(-1)^n = 0, 2, 0, 2, 0 \ldots$
DOES NOT EXIST!

DEFN IF $\lim\limits_{n\to\infty} a_n = a$, THEN

WE SAY THE SEQUENCE IS CONVERGENT. IF NOT, SEQUENCE IS DIVERGENT. (a IS REAL)

THEM:

IF $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$

WITH a & b REAL, THEN

$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

$= a + b$

PROOF:

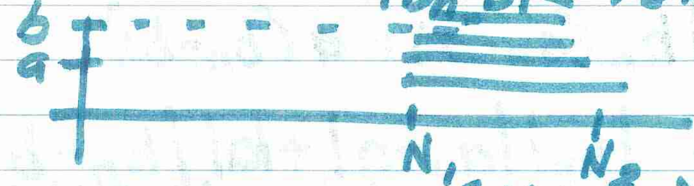
LET $\epsilon > 0$ BE GIVEN

$\lim_{n\to\infty} a_n = a \Rightarrow \exists\, N_1(\epsilon) \ni$

$|a_n - a| < \epsilon/2$ FOR $n > N_1$

$\lim_{n\to\infty} b_n\ b \Rightarrow \exists\, N_2(\epsilon) \ni$

$|b_n - b| < \epsilon/2$ FOR $n > N_2$



LET $N_3 = \max(N_1, N_2)$

FOR $N > N_3$, $|a_n + b_n - (a+b)|$

$= |(a_n - a) + (b_n - b)|$

$< |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$

FOR $n > N_3$

$\therefore \lim_{n\to\infty} (a_n + b_n) = a + b$

$$\lim_{n\to\infty} \left(\sqrt{n^2+1} - n\right) = 0$$

THEM IF $\lim_{n\to\infty} a_n = a$ ∤ $\lim_{n\to\infty} b_n = b$

THEN $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$

$= ab$ (a ∤ b REAL)

PROOF: NOTE $|a_n b_n - ab|$

$= |a_n b_n - ab_n + ab_n - ab|$

$= |b_n(a_n - a) + a(b_n - b)|$

$\leq |b_n||a_n - a| + |a||b_n - b|$

LET $\varepsilon > 0$ $\exists$ $N_2(\varepsilon) \ni |b_n - b| < \varepsilon/2|a|$

LEMMA

IF (C_n) CONVERGES, THEN IT IS BOUNDED, ie

$\exists\, P > 0 \ni |C_n| \leq P \; \forall n$

b_n CONV $\Rightarrow$ b_n BOUNDED $\therefore \exists P \ni |b_n|$
$\leq P \; \forall n$

$\lim_{n\to\infty} a_n = a \Rightarrow$ GIVEN $\varepsilon > 0$ $\exists$

$N_1(\varepsilon) \ni |a_n - a| < \varepsilon/2P$

$\therefore$ FOR $N > \text{MAX}(N_1, N_2)$

$|a_n b_n - ab| < |b_n|\varepsilon/2P + |a|\,\varepsilon/2|a| \leq \varepsilon$

QED

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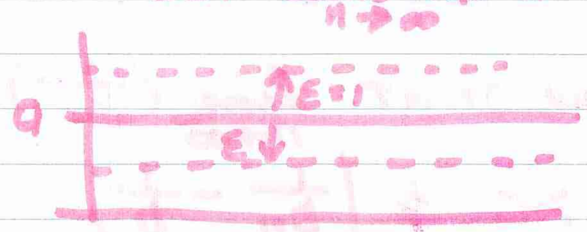
<u>LEMMA:</u> (FINDING A BOUND)

IF $\{a_n\}$ IS A CONVERGENT SEQUENCE,

$\exists\, P>0 \ni |a_n| \leq P \ \forall n$

PROOF: LET $\lim\limits_{n\to\infty} a_n = a$ (FINITE)



LET $\varepsilon = 1 \therefore \exists\, N \ni |a_n - a| < 1 \ \forall n \geq N$

$\therefore a-1 < a_n < a+1 \ \forall n \geq N$

$a_n \leq \max\{|a+1|, |a-1|\}$

$\forall n \geq N$

LET $P = \max\underbrace{\{|a_1|, |a_2| \ldots |a_N|, |a+1|, |a-1|\}}_{\text{FINITE SET OF \#'S}}$

$\therefore$ BY CONST. $|a_n| \leq P \ \forall n$ QED

<u>THEM:</u> IF $\lim\limits_{n\to\infty} a_n = a$, $\lim\limits_{n\to\infty} b_n = b$ $(b \neq 0)$

THEN:

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = \frac{a}{b}$$

(CONT.)

PROOF:

$\frac{a_n}{b_n} = a_n c_n$ WHERE $c_n = \frac{1}{b_n}$

IF WE CAN SHOW $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$;

~~THEN~~ WE'RE DONE

USING PRODUCT OF LIMITS THEM.

TO SHOW THAT $\lim_{n\to\infty} \left(\frac{1}{b_n}\right) = \frac{1}{b}$,

LET $\varepsilon > 0 \ni \left|\frac{1}{b_n} - \frac{1}{b}\right|$

$= \left|\frac{b - b_n}{b\, b_n}\right| = \frac{1}{|b|}|b - b_n| \frac{1}{|b_n|}$

ASSUME $b > 0$.

TO SHOW THAT $\frac{1}{|b_n|} < P$ FOR $n \geq$ SOME N.

FOR $\varepsilon < 0 \ \exists N_1 \ni |b_n - b| < \varepsilon$

$\therefore \varepsilon - b < b_n < \varepsilon + b$

PICK $\varepsilon = |b|/2$

$\therefore$ FOR $n > N_1$ $(b/2)$

$\cancel{-b/2 < b_n < (3/2)b}$

$\therefore \cancel{|b_n| < \frac{3b}{2} \neq 0}$

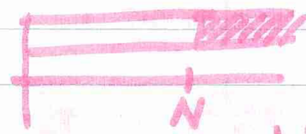
$\therefore \cancel{\text{FOR } n > N_1 \ \frac{1}{|b_n|}}$

$\frac{b}{2} < b_n < \frac{3b}{2}$

$b > 0 \Rightarrow b_n > \frac{b}{2} > 0 \ \forall n > N_1$

$\therefore$ FOR $n > N_1$

$$\frac{1}{|b_n|} < \frac{2}{|b|}$$



NOW LET $\varepsilon_1 = \left|\frac{b^2}{2}\right|\varepsilon$

$\therefore \exists N_2 \ni |b - b_n| < \varepsilon_1$ FOR $n \geq N_2$

$$\therefore \left|\frac{1}{b_n} - \frac{1}{b}\right| \leq \frac{1}{|b|}|b_n - b_n|\left|\frac{1}{b_n}\right|$$

$$< \frac{1}{|b|}\frac{|b^2|\varepsilon}{2}\frac{2}{|b|}$$

FOR $n > \text{MAX}\{N_1, N_2\}$

SO IT HAS BEEN PROVED

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$$

THRM: $\lim_{n\to\infty} k a_n = k \lim_{n\to\infty} a_n$

PROOF: LET $k = b_n$

THRM: LET $\{a_n\}$ BE A SEQUENCE OF NON-ZERO #S, THEN

$$\lim_{n\to\infty} a_n = \infty = \lim_{n\to\infty} \frac{1}{a_n} = 0$$

(CONT.)

PROOF: LET $M>0$

$$\lim_{n\to\infty} a_n = \infty \Leftrightarrow \lim_{n\to\infty} |a_n| = +\infty$$

$\Leftrightarrow$ ($M>0$, HOWEVER LARGE $\exists$ $N \ni n>N \Rightarrow |a_n|>M$)

$\Leftrightarrow$ FOR $n>N$, $\frac{1}{|a_n|} < \frac{1}{M} = \varepsilon$

$\Leftrightarrow \lim_{n\to\infty} \frac{1}{a_n} = 0$

__THEM__: LET $a>1$

$$\lim_{n\to\infty} a^n = +\infty$$

__PROOF__: LET $p = a-1$

$\therefore a = 1+p \;\therefore a^n = (1+p)^n$

$= 1+np+\frac{n(n-1)}{2}p^2 + \dots + p^n$

$p>0$

$\Rightarrow a^n = (1+p)^n > (1+np)$

LET $B>0$ (LARGE)

$a_n > B$ IF $(1+np > B)$

$\therefore n > (B-1)/p$

$\therefore$ FOR $n >$ ANY INT LARGER THAN $(B-1)/p$,

$a^n > B$, qed

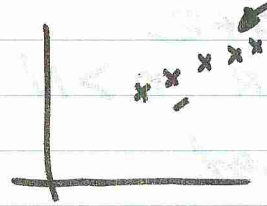
CORROLARY: $\lim_{n\to\infty} a^n = 0$ IF $0<a<1$

$b = \frac{1}{a} > 1 \therefore \lim_{n\to\infty} b^n = \infty$

$\Rightarrow \lim_{n\to\infty} \frac{1}{a^n} = \infty$

$\therefore \lim_{n\to\infty} a_n = 0$ (THEM XI)

9-24-70



ALWAY INCREASES OR STAY THE SAME (MONOTONE SEQUENCE)

a_n

THEM IF $\{a_n\}$ IS A MONOTONE SEQUENCE BOUNDED ABOVE, THEN $\lim_{n\to\infty} a_n$ EXISTS & IS FINITE

(BY MONOTONE $\Rightarrow a_n \leq a_n+1 \ \forall \cdot n.$)

(BOUNDED ABOVE $\Rightarrow \exists P \ni a_n \leq P \ \forall \ n \in PI$)

PROOF: LET $S = \{a_n \mid n \in PI\}$

S IS BOUND $\therefore$ S HAS L.U.B. $(=a)$

WE WILL PROVE THAT $\lim_{n\to\infty} a_n = a$

LET $\varepsilon > 0$

$a_n \leq a \ \forall n$ – DEF. OF L.U.B.

$a = a + \varepsilon \ \forall \ \varepsilon > 0$

$\therefore a_n < a + \varepsilon \ \forall \ n \in P.I.$

(CONT.)

TO SHOW $\exists N \ni a_n > a - \varepsilon$ FOR $n > N$

ASSUME $a_n \le a - \varepsilon \ \forall n$

LET $\varepsilon = \frac{a}{2} \Rightarrow a_n \le a_n - a/2 = a/2$

*CONTRADICTS FACT THAT

a IS L.U.B.

$\therefore \exists N(\varepsilon) \ni a_N > a - \varepsilon$

BUT $a_N \le a_n$

$\forall n > N$ (MONOTONISTICALLY)

$\therefore a_n \ge a_N > a - \varepsilon \ \forall n > N$

$\therefore$ ~~a_n~~ $a - \varepsilon < a_n < a + \varepsilon$

ie $|a_n - a| < \varepsilon$ FOR $n > N$

$\therefore \lim_{n \to \infty} a_n = a$

qed

EX $a_n = \frac{1}{n}$ (HARMONIC SEQUENCE)

BOUNDED BELOW BY 0

$\frac{1}{n} > 0 \ \forall n$

ALSO $n+1 > n \ \forall n$ $\frac{1}{n} > \frac{1}{n+1}$

$\therefore a_n > a_{n+1}$

$\therefore \lim_{n \to \infty} \frac{1}{n}$ EXISTS

QUESTION

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2.718\ldots = e$$

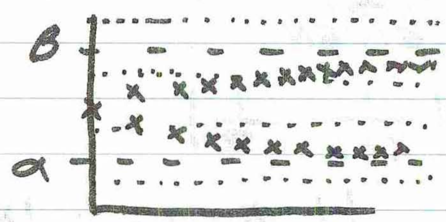
FACTS KNOWN:

$\left(1 + \frac{1}{n}\right)^n < 3 \ \forall n$

$\ni \left(1 + \frac{1}{n}\right)^n \le \left(1 + \frac{1}{n+1}\right)^{n+1} \forall n$

PROBLEM HINTS:

Pg 33 #1



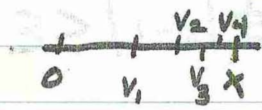
~~?~~ ANY SUBSEQUENCE CONVERGES TO THE LIMIT OF THE SEQUENCE. (COUNTERPOSITIVE OF THEM. IV)

#2) BOUND SEQUENCE IS NOT BOUNDED

0, 1, 0, 1, 0, 1 . . . .

#10) $(-1)^n$ AND $|(-1)^n|$

$\downarrow$ DIVER $\quad\quad \downarrow$ CON

#15) 

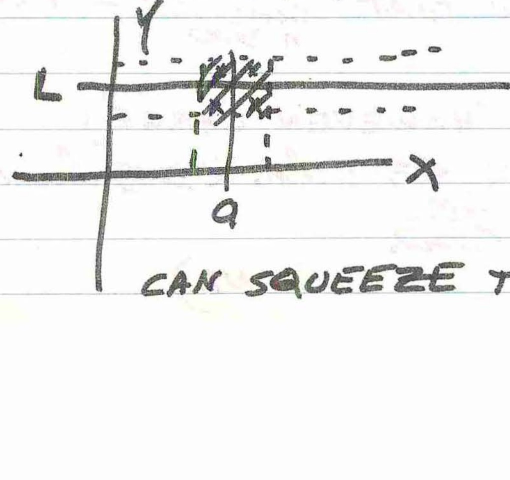
ETC.

- -

Let $f(x)$ BE A FUNCTION (DEFINED IN SOME NEIGHBORHOOD OF a)

WE SAY THAT THE LIMIT AS $x \to a$ OF $f(x)$ IS L, IF $\forall \epsilon > 0 \; \exists \; \delta(a, \epsilon) \ni 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$, ∴ WE WRITE

$$\lim_{x \to a} f(x) = L$$



CAN SQUEEZE THE BOX

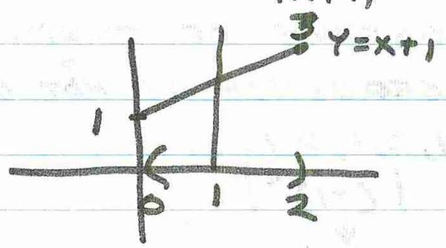
EX. TO SHOW $\lim\limits_{x \to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$

Let $\varepsilon > 0$ BE GIVEN

$$\left| \frac{x-1}{x^2-1} - \frac{1}{2} \right| = \left| \frac{1}{x+1} - \frac{1}{2} \right|$$

$$= \frac{2-(x+1)}{2(x+1)} = \left| \frac{1-x}{2(x+1)} \right| = \frac{1}{2} \frac{|x-1|}{|x+1|}$$

OR $|f(x)-1| = \frac{1}{2} \frac{|x-1|}{|x+1|}$



$|x+1| > 1$

FOR $|x-1| < 1$

$$\therefore \frac{1}{2} \frac{|x-1|}{|x+1|} < \frac{1}{2}|x-1| < \varepsilon$$

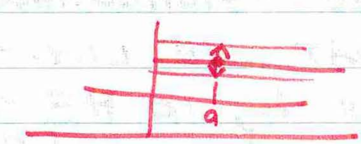
$$\therefore \delta = (2\varepsilon, 1)$$

WHICHEVER IS SMALLER

9-25-70

QUESTIONS

$\varepsilon > 0$



DIVERGENCE:

EX) $a_n = 2^n$

TO SHOW THAT $\lim\limits_{n \to \infty} 2^n = +\infty$

$\therefore M > 0$ (HOWEVER LARGE)

$2^n > M \iff n \ln 2 > \ln M$

$\iff n > \frac{\ln M}{\ln 2}$

(CONT.)

$\therefore$ LET $N = \left\{ \begin{array}{c} \text{SMALLEST} \\ \text{INT LARGER} \\ \text{THAN } M/\ln 2 \end{array} \right\}$

$\therefore$ FOR $n > N$. $2^n \geq M$

DON'T CONFUSE SERIES (Σ) & ~~[illegible]~~ SEQENCE

DEFN. LET $\{a_n\}$ BE A SEQ, THEN $\sum_{n=1}^{\infty} a_n$ IS AN INFINITE SERIES.

THEM

$$\sum_{K=1}^{\infty} a_K = \lim_{n \to \infty} S_n$$

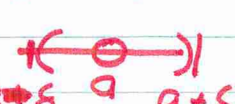
$$S_n = \sum_{K=1}^{n} a_K$$

n	a_n	S_n
1	1	1
2	$\frac{1}{2}$	$1 + \frac{1}{2} = 3/2$
3	$\frac{1}{3}$	$\frac{3}{2} + \frac{2}{3} = 1\frac{5}{6}$
4	$\frac{1}{4}$	$\vdots$
5	$\frac{1}{5}$	$\vdots$

DEFN. $\lim_{x \to a} f(x) = L$ IF $\forall\, \varepsilon > 0\ \exists\ \delta(a, \varepsilon) \ni$

$$\underbrace{0 < |x-a| < \delta}_{\text{A DELETED NEIGHBORHOOD}} \Rightarrow |f(x) - L| < \varepsilon$$

A DELETED NEIGHBORHOOD $\rightarrow$

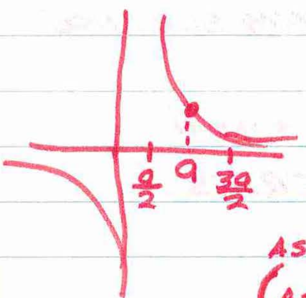


$$|f(x) - L| = |x - a| a < \varepsilon \Rightarrow \delta < \frac{\varepsilon}{a}$$

(CONT.)

EX) $\lim\limits_{x \to a} \frac{1}{x} = \frac{1}{a}$ PROV. $a \neq 0$

let $\varepsilon > 0$, $\left|\frac{1}{x} - \frac{1}{a}\right| = \left|\frac{a-x}{ax}\right|$



$= |x-a| \left|\frac{1}{ax}\right|$ ← a TERM

FOR THE SAKE OF ARGUMENT,
ASSUME $\frac{a}{2} < x < \frac{3a}{2}$ $\therefore$ $\frac{1}{x} < \frac{2}{a}$
(ASSUMED δ = AT MOST $\frac{a}{2}$)

$\therefore |x-a| \frac{1}{|ax|} < \underbrace{|x-a| \frac{1}{|a|} \frac{2}{|a|}}_{\text{IF } |x-a| < a/2} < \varepsilon$

$\Rightarrow |x-a| < \frac{\varepsilon a^2}{2}$ $\Leftarrow$ OTHER δ)

SO $\delta = \text{MIN} \left\{ \frac{|a|}{2}, \frac{\varepsilon a^2}{2} \right\}$

- - - - - - - - - - - - -

$\delta(a, \varepsilon) \Rightarrow \delta = f(a, \varepsilon)$

- - - - - - - - - - - - - - - - -

$|x-a| < ? \Rightarrow \left|\frac{1}{x} - \frac{1}{a}\right| < .1$

? = .45 FROM PREVIOUS EX.

9-28-70

IF HE GIVES US THE ε, YOU MUST BE ABLE TO GIVE ME AN δ IN 10 OR LESS SECONDS (SO WHAT)

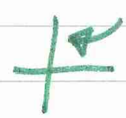
$$\lim_{x \to a} f(x) = L \qquad -\infty \leq a \leq \infty$$

IF $\forall\, \varepsilon > 0\ \exists\ \delta(a,\varepsilon) \ni 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

CAN BE 1 SIDED, OR 2 SIDED

ONE SIDED LIMIT (a TO a+δ

<u>DEF</u> $\lim_{x \to a+} f(x) = L$ IF $\forall\, \varepsilon > 0\ \exists\ \delta(a,\varepsilon)$



$\ni\ 0 < x - a < \delta$

$\Downarrow$

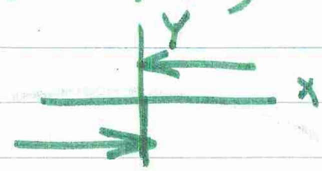
(OR $x \in (a, \text{to } a+\delta)) \Rightarrow |f(x) - L| < \varepsilon$

<u>SIMILARLY</u> $\lim_{x \to a^-} = L$ IF $\forall\, \varepsilon > 0\ \exists\ \delta(a,\varepsilon)$

$\ni\ -\delta < x - a < 0$

$\Rightarrow |f(x) - L| < \varepsilon$

<u>EX</u> $f(x) = \dfrac{x}{|x|}\ ;\ x \neq 0 \qquad D = \{x, | x \neq 0\}$



(CONT.)

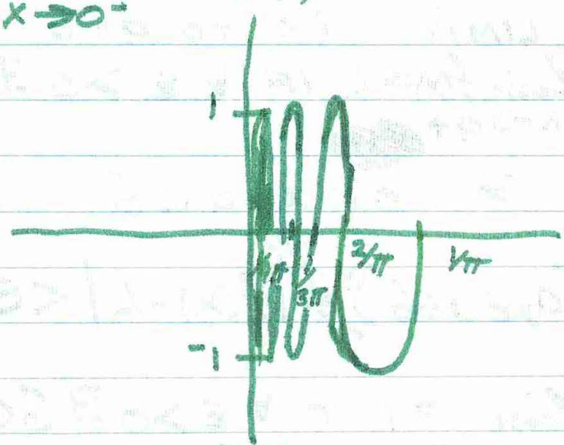
$\lim_{x \to 0} f(x)$ DOES NOT EXIST

$\lim_{x \to 0+} f(x) = 1$; $\lim_{x \to 0-} f(x) = {}^{-}1$

EX $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

$\lim_{x \to 0+} \sin\left(\frac{1}{x}\right)$ NONE EXIST

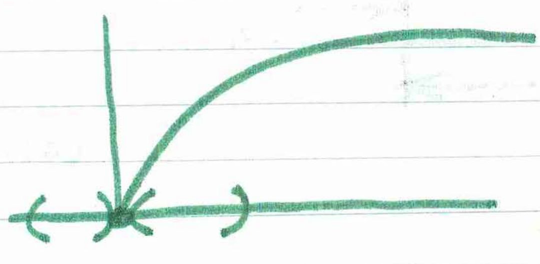
$\lim_{x \to 0-} \sin\left(\frac{1}{x}\right)$



EX $\lim_{x \to 0} \sqrt{x} = 0$ ⇐ FITS DEFINITION (POINT IN NEIGHBORHOOD)

$\lim_{x \to 0+} \sqrt{x} = 0$

$\lim_{x \to 0-} \sqrt{x}$ = UNDEFINED



DEFINE INFINITE LIMITS

1) $\lim_{x \to a} f(x) = +\infty$ IF $\forall\ M > 0$

(HOWEVER LARGE) $\exists\ \delta(M, a) \ni$
$0 < |x - a| < \delta \Rightarrow f(x) > M$

2) $\lim_{x \to a} f(x) = -\infty$ IF $\forall\ M < 0$,

$\exists\ \delta(a, M) \ni 0 < |x - a| < \delta$
$\Rightarrow f(x) < -M$

3) $\lim_{x \to a} f(x) = \infty$

IFF $\lim_{x \to a} |f(x)| = +\infty$

EX $\lim_{x \to 0} \frac{1}{x} = \infty$

$\lim_{x \to 0^+} \frac{1}{x} = +\infty$

$\lim_{x \to \frac{\pi}{2}^-} \tan x = +\infty$

DEFN.

$\lim_{x \to +\infty} f(x) = L$ IF
$\forall \varepsilon > 0\ \exists\ M \ni x > M \Rightarrow |f(x) - L| < \varepsilon$

$\lim_{x \to \infty} f(x) = L$

IF $\forall\ M > 0$ (HOWEVER BIG) $\exists\ P \ni$
$x > P \Rightarrow f(x) > M$

DEFN.

$$\lim_{x \to t} f(x) = S \begin{array}{l} +\infty \\ -\infty \\ \text{REAL} \end{array}$$

IF $\forall$ N NBHD OF t $\exists$ n LABELED OF $S \ni$ 'BLAH BLAH BLAH'

9-29-70

~~XXX~~ IF $\lim_{x \to a} f(x) = L_1$

$\lim_{x \to a} g(x) = L$

THEN $\lim_{x \to a} \left(g(x) f(x) \right) = L_1 L_2$

PROOF: LET $\varepsilon > 0$, TO FIND A $\delta(a, \varepsilon)$ ~~X~~ $\ni 0 < |x-a| < \delta$
$\Rightarrow |f(x) g(x) - L_1 L_2| < \delta.$

$$|f(x)g(x) - L_1 L_2| = |f(x)g(x) - g(x)L_1 + g(x)L_1 - L_1 L_2|$$

$$= |g(x)(f(x) - L_1) + L_1(g(x) - L_2)|$$

$$\leq \underline{|g(x)|\,|f(x) - L_1|} + |L|_1 |g(x) - L_2|$$

$\exists\ \delta \ni |g(x) - L_2| < L_2/2 \quad (L_2 > 0)$

WHENEVER $0 < |x-a| < \delta$

$\therefore L_2/2 < g(x) < 3L_2/2$

IN ANY EVENT, FOR $0 < |x-a| < \delta$

$|g(x)| < 3L_2/2$

$\varepsilon > 0$, GIVEN, $\exists$ δ_2, δ_3

$|f(x) - L_1| < \varepsilon/3L_2$ WHEN $0 < |x-a| < \delta_2$

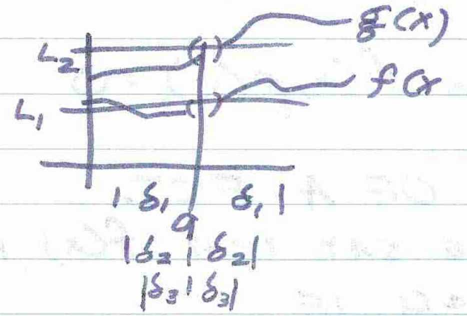
$|g(x) - L_2| < \varepsilon/2|L_1|$ WHEN $0 < |x-a| < \delta_3$

$\therefore$ LET $\delta = \min(\delta_1, \delta_2, \delta_3)$

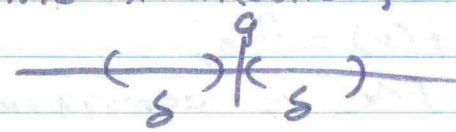
FOR $0 < |x-a| < \delta$

$|f(x)g(x) - L_1L_2| < \frac{3L_2}{2}\frac{\varepsilon}{3L_2} + \frac{|L_1|\varepsilon}{2L_1} < \varepsilon$

QED



δ LIMITS x AROUND a



PROVE: $\lim\limits_{x \to a}(96x^{14} + 14x^{12} - 72.7x^{11} + \ldots + 90{,}143x + 1) = A$

WHERE $A = 96a^{14} + 14a^{12} + \ldots\ldots + 1$

<u>LEMMA</u>: $\lim\limits_{x \to a} x = a$ LET $\varepsilon > 0$

$|x-a| < \varepsilon$ IF $|x-a| < \varepsilon \therefore \delta = \varepsilon$

<u>LEMMA</u>: $\lim x^n = a^n \; \forall n \in P.I.$

S_n: $\lim\limits_{x \to a} x^n = a^n$ (BY INDUCTION)

S_1 : TRUE FOR LEMMA

ASSUME $\lim\limits_{x \to a} x^n = a^n$

THEN $\lim\limits_{x \to a} x^{n+1} = \lim x^n x$

$= \lim x^{n+1}$

$= a^{n+1}$

QED

THEM: $\lim_{x \to a} (b_n x^n + b_{n-1} x^{n-1} + \ldots + b_0)$

$$\equiv (b_n a^n + \ldots + b_1 a + b_0)$$

LIM. OF SUM = SUM OF LIM $(\lim cf = c \lim f)$

THEM: $f(x)$ IS BOUNDED @ $x = a$ &

$\lim_{x \to a} g(x) = 0$

then $\lim f(x) g(x) = 0$

~~PROOF~~//

CONTINUITY OF A FCT.

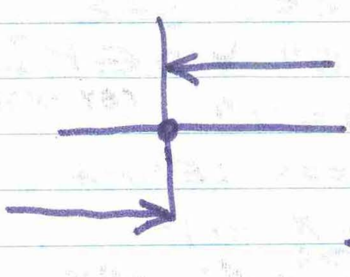
DEF: WE SAY THAT $f(x)$ IS CONTINUOUS AT $x = a$ IF

1) a IS IN DOMAIN OF f $(a \in \mathcal{D}(f))$

2) $\lim_{x \to a} f(x) = f(a)$

DEF: IF $f(x)$ IS CONTINUOUS $\forall x_0 \in A$, THEN WE SAY THAT $f(x)$ IS CONTINUOUS ON THE SET A

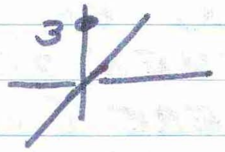
EX) $\text{sgn}(x) \begin{cases} +1, & x > 0 \\ -1, & x < 0 \\ 0, & x = 0 \end{cases}$

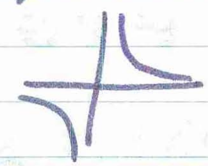


DISCONTINUOUS AT $x = 0$

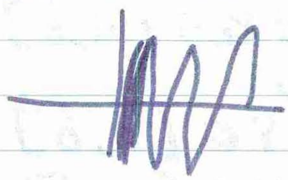
$\lim_{x \to 0} \text{sgn}\, x \neq \text{sgn}\, 0 = 0$

EX) 1) $f(x) = \begin{cases} 2x, & x \neq 0 \\ 3, & x = 0 \end{cases}$



2)  $\frac{1}{x}$

INFINITE

3)  $\sin \frac{1}{x}$

NO LIM. EXIST

4) $f(x) \begin{cases} = \frac{1}{q}; & p/q \text{ ~~REAL~~ RATIONAL} \\ = 0; & x \text{ IS IRR.} \end{cases}$

10-2-70

MORE IRRATIONALS THAN RATIONALS

$$\begin{array}{l} 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots \\ 2, \frac{2}{2}, \frac{2}{3}, \frac{2}{4}, \frac{2}{5} \ldots \\ 3, 3/2, 3/3, 3/4, 3/5 \ldots \end{array}$$

1 TO 1 CORRESPONDANCE BETWEEN RATIONALS AND INTEGERS

10-5-70

THEM: EVERY BOUNDED SEQUENCE HAS A CONVERGENT SUBSEQ.

THEM: Let $f(x)$ BE CONTINUOUS ON $[a,b]$, THEN $f(x)$ IS BOUNDED THERE (ie $\exists M \ni |f(x)| \leq M \; \forall x \in [a,b]$

PROOF: ASSUME $f(x)$ IS NOT BOUNDED

$\therefore \forall n \in PI \; \exists x_n \in [a,b] \ni |f(x_n)| > n$

OR $M > 0 \; \exists x \in [a,b] \ni |f(x)| \geq M$

$1 > 0 \; \exists x \in [a,b] \ni |f(x_1)| > 1$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

NOW $\{x_n\} \subseteq [a,b]$

$\therefore \{x_n\}$ IS BOUNDED

$\exists$ A SUBSEQ OF $\{x_{n_k}\}$ WHICH IS CONVERGENT

Let x_0 BE THE LIMIT OF $\{x_{n_k}\}$

ie) $\lim\limits_{k\to\infty}\{x_{n_k}\} = x_0$

$\lim\limits_{k=\infty} x_{n_k} = x_0$

BUT $f(x)$ IS CONT.

$\Rightarrow \lim\limits_{k\to\infty} f(x_{n_k}) = f(\lim\limits_{k\to\infty} x_{n_k}) = f(x_0)$

$\ni x_0 \in [a,b]$

BUT: $\{f(x_{n_k})\} \to f(x_0) \Rightarrow \forall \varepsilon > 0$

$N \ni k > N \; \ni$ $|f(x_{nk}) - f(x_0)| < \varepsilon$

$f(x_0) - \varepsilon < f(x_{n_k}) < f(x_0) + \varepsilon$

CONTRADICTS THE CONSTUCTION

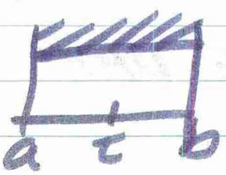
(CONT.)

OF $\{x_n\}$ SINCE $\{x_{n_k}\}$ IS A SUB SEQ. OF $\{x_n\}$

QED

THEM: EVERY $f(t)$ CONTINUOUS AN $[a,b]$ IS BOUNDED

THEM: IF $f(t)$ IS CONT. ON $[a,b]$ $\exists$ $t \in [a,b] \ni f(x) \leq f(t)$ $\forall x \in [a,b]$ ie) A CONTINUOUS FUNCTION ON A CLOSED INTERVAL TAKES ON ITS MAX.



PROOF: $f(x)$ IS CONT. ON $[a,b]$ $\therefore$ BDED. AN $[a,b]$ (PREV THEM.)

$\therefore \{f(x) \mid x \in [a,b]\}$ IS A BDED SET OF REAL #'s.

SINCE EVERY BOUND SET OF #'s HAS A L.U.B., LET $M = L.U.B\ \{f(x) \mid x \in [a,b]\}$

$\therefore \forall \varepsilon > 0\ \exists\, q \ni |f(q) - M| < \varepsilon$

~~WHEN~~

$\Rightarrow \varepsilon = \frac{1}{n}\ \exists\, x_n \ni |f(x_n) - M| < \frac{1}{n}$

$\{x_n\}$ IS A SEQ $\ni$ $\{x_n\} \subseteq [a,b]$, AND IS THUS BOUNDED. EVERY BNDED SEQ HAS A CONV. SUBSEQUENCE

(CONT.)

$\exists \{x_{n_k}\} \subseteq \{x_n\} \ni \lim\limits_{k\to\infty} x_{n_k}$ EXISTS

LET $t = \lim\limits_{k\to\infty} x_{k_n}$

TO SHOW $f(t) \overset{?}{=} M$

1) $\lim\limits_{k\to\infty} f(x_{n_k}) = f(\lim\limits_{k\to\infty} x_{n_k}) = f(t)$

BECAUSE $f(t)$ IS CONTINUOUS

2) $\lim\limits_{n\to\infty} f(x_n) = M$, BY CONST

3) BUT A SUB SEQ. OF A CONV. SEQ CONV. TO THE SAME LIMIT AS THE SEQ:

$f(x_n)$, $f(x_{n_k})$ → SAME LIMIT

10-6-70

Thm: If $f(x)$ IS CONT FOR $x \in [a,b]$ ∋ IF $f(a)f(b) < 0$. THEN $\exists\ x_0 \in [a,b] \ni f(x_0) = 0$

PROOF: WITHOUT LOSS OF GENERAL $f(a) < 0$ (ie $(f(b) > 0)$)

Let $A = \{x \mid f(x) < 0\}$

CLEARLY $A \neq \emptyset$ FOR $a \in A$; $A \subseteq [a,b]$ ∴ A IS BOUNDED SINCE EVERY BOUNDED SET OF REAL #'S HAS A L.U.B., Let $x_0 = L.U.B.\ A$

CLEARLY $f(x_0) = 0$

SHOW $f(x_0) \not\equiv 0$

ASSUME $f(x_0) \neq 0$

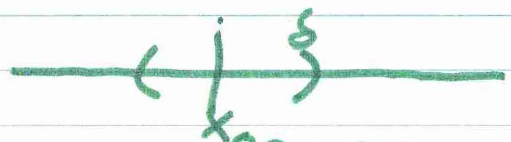
~~FOR SAKE OF ARGUMENT~~

(i) $f(x_0) < 0 \Rightarrow$ LET $\varepsilon > 0$, BY CONT. $\exists\ \delta > 0 \ni |x - x_0| < \delta$
$\Rightarrow |f(x) - f(x_0)| < \varepsilon$

ie) $|x - x_0| < \delta \Rightarrow f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$

PICK $\varepsilon = -\tfrac{1}{2} f(x_0) > 0$

$\therefore |x - x_0| < \delta \Rightarrow$
$\tfrac{3}{2} f(x_0) < f(x) < \tfrac{1}{2} f(x_0)$



$\exists x_{\ni} x < x_0 \ni f(x) < 0$

$\therefore x_0 \neq L.U.B$ (IMPOSSIBLE)

(ii) SIMILAR DISPROVAL OF $f(x_0) > 0$

$\therefore f(x_0) \equiv 0$

THEM: Let $f(x)$ BE CONTINUOUS FOR $x \in [a, b]$

THEN $f(x)$ ASSUMES EVERY VALUE BETWEEN $f(a)$ & $f(b)$ AT LEAST ONCE

PROOF. W.L.O.G. ASSUME $f(b) < f(a)$

LET $M \in [f(b), f(a)]$

LET $\phi(x) = f(x) - M$

$\phi(b) = f(b) - M < 0$

$\phi(a) = f(a) - M > 0$

$\therefore$ BY PREVIOUS THEOREM

$\exists\, x_0 \in [a, b] \ni \phi(x_0) = 0 = f(x_0) - (M)$ Q.E.D.

10-10-70

UNIFORM CONTINUITY

AN INTERVAL PROPERTY

WE SAY THAT $f(x)$ IS UNIFORM ON $[a, b]$ IF $\forall\, \varepsilon > 0\ \exists\ \delta(\varepsilon) \ni \forall\, x_0 \in [a, b]$

$0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \delta$

<u>THEM</u>: IF $f(x)$ IS CONTINUOUS ON $[a, b]$, IT IS UNIFORM. CONT. ON $[a, b]$

(CONTINUITY ON A CLOSED INTERVAL $\Rightarrow$ UNIFORM CONT.)

1) IF $0 < |x_0 - x| < \delta$, THEN $|f(x_0) - f(x)| < \varepsilon$

$|L - f(x)| \le K|x_0 - x| < \varepsilon$; $\delta = \varepsilon/K$

2) $|x' - x''| < \delta$, THEN $|f(x') - f(x'')| < \varepsilon$

$|f(x') - f(x'')| \le K|x' - x''| < \varepsilon$; $\delta = \varepsilon/K$

PROOF: SUPPOSE $f(x)$ IS NOT UNIFORMLY CONTINUOUS. ON $[a, b]$

FOR ANY δ $\exists$ $x_1, x_2 \ni$

$|f(x_1) - f(x_2)| \ge \varepsilon$,

BUT $|x_1 - x_2| < \delta$

IN PARTICULAR, FOR $\delta = 1/2$

$\exists$ PTS $x_n', x_n'' \ni |x_n' - x_n''| < 1/n$

BUT $|f(x_n') - f(x_n'')| \ge \varepsilon$

x_n' IS A BOUNDED SEQ. FOR $[a, b]$ $\therefore$ $\exists$ A CONVERGING SUBSEQUENCE $\{x'_{n_k}\}$

let $x_0 = \lim\limits_{n \to \infty} x'_{n_k}$

BY CONTONUITY OF $f(x)$,

$\lim\limits_{k \to \infty} f(x'_{n_k}) = f(\lim\limits_{k \to \infty} x_{n_k}) = f(x_0)$

KNOW FOR TEST

THEM	PG	ALL DEF.
I	19	
V	29	
VIII	30	
XII	32	
V	38	
IV	46	
I	48	
II	48	
III	48	

10-13-70

DEFN LET $Y=f(x)$ WE SAY THAT $\left(\frac{dy}{dx}\right)_{x_0} = \lim\limits_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$

IS THE DERIVITIVE OF Y @ $X=X_0$, PROVIDED THE LIMIT EXITS.

$$\left(\frac{dy}{dx}\right)_{x_0} = Y'(x_0) \quad \text{(A NUMBER)}$$

IF WE WRITE $\frac{dy}{dx}$ OR Y', WE MEAN THE FUNCTION OBTAINED BY TAKING THE DERIVITIVE A EACH PT X OF SOME SET A

i.e.) $\frac{dy}{dx} = \lim\limits_{\Delta x \to 0} \frac{Y(x+\Delta x)-Y(x)}{\Delta x}$

EX) $Y = x^2$

$$\frac{dy}{dx} = 2x = 6(x) \left(\frac{dy}{dx}\right)_3 = 6$$

$$\left(\frac{dy}{dx}\right)_\alpha = 2\alpha$$

(THE DER FROM THE RT.)

~~THE RT. HAND DER. @ $X=X_0$ IS~~

$$\left(\frac{dy}{dx}\right)^+_{x_0} = \lim\limits_{x \to 0^+} \frac{Y(x_0+\Delta x)-Y(x_0)}{\Delta x}$$

(RT HAND DER.)

$$\lim_{\Delta x \to 0^+} \frac{Y(x_0 + \Delta x) - Y(x_0^-)}{\Delta x} \quad \text{OR} \quad Y(x_0^-) = \lim_{x \to x_0^-} Y(x)$$

EX) $Y = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$

RT. HAND DER. ALLOWS LIMIT NOT TO BE IN THE DOMAIN

THEM IF $Y = f(x)$ IS DIFFERENTIABLE @ $x = x_0$, THEN IT IS CONT (ie DIFF $\Rightarrow$ CONT.) (NOT CONVERSE)

PROOF:

LET $Y = f(x)$ BE DIFFERENTIABLE @ $x = x_0$

$$\therefore \left(\frac{dy}{dx}\right)_{x_0} = \lim_{\Delta x \to 0} \frac{Y(x + \Delta x) - Y(x_0)}{\Delta x}$$

EXISTS.

(1) $x_0 \in D(f)$, WHY FOR $Y = f(x)$ TO ~~HAV~~ BE DIFF @ $x = x_0$, $Y(x_0)$ MUST EXIST & $Y(x_0) = f(x_0)$

(2) TO SHOW $f(x)$ IS CONT., WE MUST SHOW THAT:

$$\lim_{x \to x_0} f(x) \equiv f(x_0) \;\&\; Y(x_0)$$

(CONT.)

NOW $f(x+\Delta x)-f(x_0) \equiv \Delta x \frac{f(x+\Delta x)-f(x_0)}{\Delta x}$, FOR $\Delta x \neq 0$

$$\lim_{x \to 0} \{f(x_0+\Delta x)-f(x_0)\} \equiv \lim_{\Delta x \to 0} \left(\Delta x \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\right)$$

$$= \lim_{\Delta x \to 0} \left(\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\right)\left(\lim_{x \to 0} \Delta x\right)$$

$\because$ ~~PROP.~~ LIM OF PROD = PROD OF LIMITS, PROVIDED BOTH LIMITS EXIST & THE FIRST EXISTS BY ASSUMPTION OF DIFFERENTIABILITY @ x_0

$$\left(\frac{dy}{dx}\right)_{x_0} \cdot 0 = 0$$

$$\therefore \lim_{\Delta x \to 0} [f(x_0+\Delta x)-f(x_0)] = 0$$

$$= \lim_{\Delta x \to 0} f(x_0+\Delta x) - f(x_0) = 0$$

$$\therefore \lim_{\Delta x \to 0} f(x_0+\Delta x) \equiv f(x_0)$$

Q.E.D.

LEMMA LET $Y(x)$ BE A DIFF. FUNCTION. FOR $a \le x \le b$

THEN FOR $E(x, \Delta x) = \frac{dY}{dx} - \frac{\Delta Y}{\Delta x}$,

$$\lim_{\Delta x \to 0} E(x, \Delta x) = 0$$

EX $Y(x) = x^2$; $\frac{dY}{dx} = 2x$

$\Delta Y = 2x(\Delta x) + (\Delta x)^2 = (x + \Delta x)^2 - x^2$

$\therefore E(x, \Delta x) = 2x - (2x + \Delta x) = -\Delta x$

CHAIN RULE

$Y = f(u)$, $y = u(x)$

Y IS DIFF @ $u_0 = u(x_0)$

$\ni$ U IS DIFF. @ x_0

CONSIDER:

$$\frac{dY}{dx} = \frac{dY}{du}\frac{du}{dx}$$

TAKE $E(u, \Delta u) = \frac{dY}{du} - \frac{\Delta Y}{\Delta u}$

MULT BY Δu

$$\Delta u E = \frac{dY \Delta u}{du} - \Delta Y$$

$$\frac{\Delta u}{\Delta x} E = \frac{dY}{du}\frac{\Delta u}{\Delta x} - \frac{\Delta Y}{\Delta x}$$

$$\lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x}\right) E(u, \Delta u) = \lim_{x \to 0} \left(\frac{dY}{du}\frac{\Delta u}{\Delta x} - \frac{\Delta Y}{\Delta x}\right)$$

$$= \frac{dY}{du}\frac{du}{dx} - \frac{dY}{dx}$$

$\Delta u \to 0$ AS $\Delta x \to 0$

~~$\frac{dY}{dx}$ ...~~.

10-19-70

EX) $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

1) FCT IS DIFF @ $x = 0$
2) IT'S DERIVITIVIVE IS NOT $\left[\frac{d}{dx}\left(x^2 \sin\left(\frac{1}{x}\right)\right)\right]_{x=0}$

2 ALTERNITIVES

$\frac{d}{dx}\, x^2 \sin \frac{1}{x}$ IS NOT CONT @ $x=0$

DEMO OF 2 & 2 ALT.

BY RULES OF DIFF:

$$\frac{d}{dx}\left(x^2 \sin \frac{1}{x}\right) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

$\sin \frac{1}{x}$ AND $\cos \frac{1}{x}$ DON'T HAVE LIMIT AS $x \to 0$

$$\left(\frac{dy}{dx}\right)_{x=0} \overset{DEF}{=} \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{y(\Delta x) - y(0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^2 \sin \frac{1}{\Delta x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \Delta x \sin \frac{1}{\Delta x} = 0$$

Q.E.D

10-20-70

MC_3 - INTERA 4
MC_4 - LAHOPITAL 3

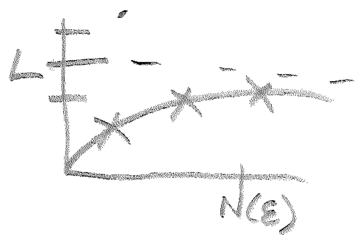
34/105

BOB MARKS

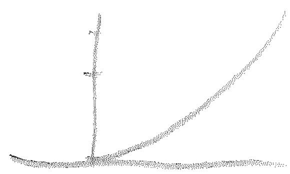
Define each of the following

(a) $\lim_{n \to \infty} a_n = L$ IFF $\forall\, \varepsilon > 0\ \exists N(\varepsilon)^{\text{e.p.I.}} \ni n > N(\varepsilon) \Rightarrow |a_n - L| < \varepsilon$

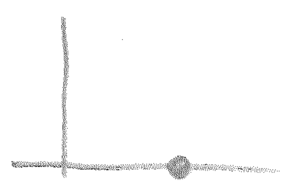
Typographical error ?



(b) $\lim_{n \to \infty} a_n = +\infty$ IFF $\forall\, B\ \exists N(B)^{\text{e.p.I}} \ni n > N(B) \Rightarrow a_n > B$



(c) $\lim_{x \to x_0^+} f(x) = L$ IFF $\forall\, \varepsilon\ \exists\ \delta(\varepsilon) > 0 \ni 0 < x - a < \delta$
$\Rightarrow |f(x) - L| < \varepsilon$



(d) $\lim_{x \to +\infty} f(x) = L$ IFF $\forall\, \varepsilon > 0\ \exists\ M(\varepsilon) \ni n > M(\varepsilon)$
$\Rightarrow |f(x) - L| < \varepsilon$

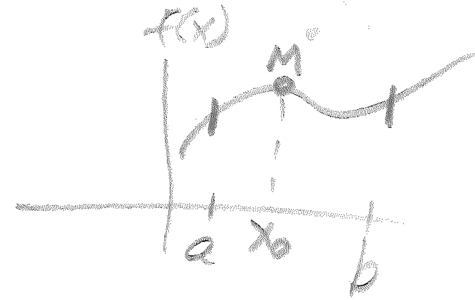
14
15
5
34

(e) $f(x)$ is continuous at $x = a$

IFF $\forall\, \varepsilon > 0\ \exists\ \delta(\varepsilon) > 0 \ni |x - a| < \delta$
$\Rightarrow |f(x) - f(a)| < \varepsilon$

need $a \in D(f)$

14



(i) $\exists M$, $f(x) \leq M$ $\forall x \in [a,b]$, and

(ii) $\exists x_0 \in [a,b] \ni f(x_0) = M$.

The statements in the proof of this theorem are given below. You are to supply the reason for each statement made.

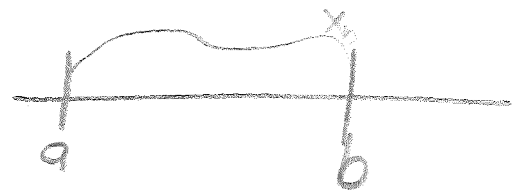
1. $\{f(x) \mid x \in [a,b]\}$ is a bounded set	ANY FUNCTION CONTINUOUS OVER A FINITE INTERVAL IS BOUNDED
2. Let $M = \text{lub}\{f(x) \mid x \in [a,b]\}$.	EVERY BOUNDED ~~FUNCTION~~ Set HAS A L.U.B.
3. $f(x) \leq M$ $\forall x \in [a,b]$. (Hence part (i)) To find x_0,	DEFINITION OF L.U.B.
4. $\forall n \in PI$, Let x_n be $\ni$ $x_n \in [a,b]$ and $\lvert f(x_n) - M \rvert < \frac{1}{n}$	
5. $\therefore \lim_{n \to \infty} f(x_n) = M$ -3	TAKING $\cancel{\lim_{x \to \infty}} \lvert f(x_n) - M \rvert = 0$ -3
6. $\{x_n\}$ is a bounded sequence. -3	ALL ~~3~~ CONV. ~~SEQ.~~ ~~ARE BOUNDED~~ but not the reason
7. $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is convergent.	EVERY BOUNDED SEQUENCE HAS A CONVER~~GENT~~ SUBSEQUENCE
8. Let $x_0 = \lim_{k \to \infty} x_{n_k}$	→ CON. SEQ. CONVERGES UNIQUELY
$\lim_{k \to \infty} f(x_{n_k}) = f(\lim_{k \to \infty} x_{n_k}) = f(x_0)$ -3	→
9. $\therefore \lim_{n \to \infty} f(x_n) = f(x_0)$	→ ~~IF~~ A SEQ. ~~CONV.~~ TO a ANY SUB-SEQ ~~CONV.~~ TO a
10. $f(x_0) = M$ why? -3	→ $\lim_{n \to \infty} f(x_n) = f(x_0) = M$

$\frac{14}{15}$
$\frac{5}{34}$

15

[illegible]

If (i) $\{x_n\} \subseteq (a,b)$,



(ii) $\lim_{n \to \infty} x_n = x_0$, Then $x_0 \in [a,b]$.

(b) Show, by giving an appropriate counter example that $x_0 \in (a,b)$ is not a necessarily correct conclusion.

4. Using the definition of the limit, prove that

$$\lim_{x \to 2} \left(\frac{x^2 - 3x}{x - 1}\right) = -2$$

(i.e. given ε, find the δ).

$\lim_{x \to a} f(x) = L$ IF $\forall \varepsilon \, \exists \, \delta(\varepsilon) > 0 \ni 0 < |x-a| < \delta$

$\Rightarrow |f(x) - L| < \varepsilon$

LET $\varepsilon > 0$

$$\left| \frac{x^2 - 3x}{x-1} + 2 \right| < \varepsilon$$

$$\left| \frac{x^2 - 3x + 2x - 2}{x-1} \right| < \varepsilon$$

$$\left| \frac{x^2 - x - 2}{x-1} \right| < \varepsilon$$

$$\left| \frac{(x+1)(x-2)}{(x-1)} \right| < \varepsilon$$

$$x > 1 \Rightarrow \left| \frac{x-2}{x-1} \right| \le \left| \frac{(x+1)(x-2)}{(x-1)} \right| < \varepsilon$$

$$\left| \frac{x-2}{x-1} \right| < \varepsilon ?$$

$$\left| \frac{x}{x-1} \right| - \left| \frac{2}{x-1} \right| \le \left| \frac{x}{x-1} - \frac{2}{x-1} \right| < \varepsilon$$

$$x > 1 \Rightarrow \left| \frac{x}{x-1} \right| - \left| \frac{2}{x-1} \right| \ge \cancel{\left| \frac{2}{x-1} \right| < \varepsilon}$$

$$\cancel{\frac{-2}{|x-1|} < \varepsilon}$$

$$\cancel{-2 < \varepsilon |x-1|}$$

5

5

V. Let S be a set of real numbers.

(a) Define what is meant by lub S

$S = \{n \mid n \in \text{REAL \#'s}\}$

IF $N = \text{L.U.B. OF } S$

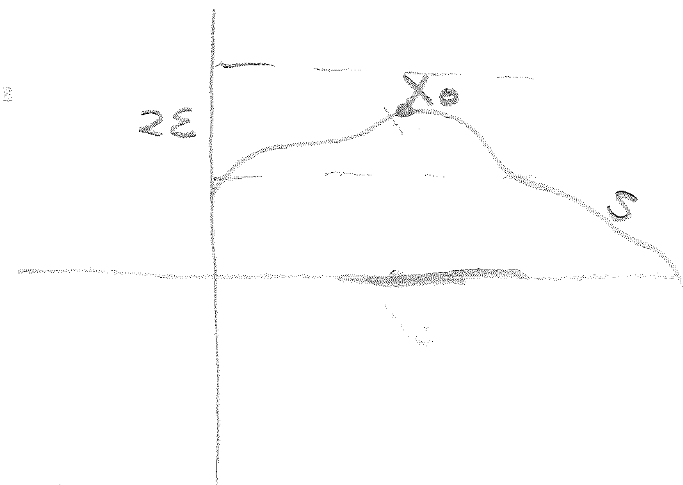
$\Rightarrow N \geq n \in S$

(b) Let $x_0 \in R \ni \forall \varepsilon > 0$

(i) $\exists\, y \in S \ni y > x_0 - \varepsilon$

(ii) $\forall\, x \in S,\ x < x_0 + \varepsilon$

Prove that $x_0 = \text{lub } S$



[illegible]

[illegible] is being taught, [illegible] the following statements will indicate [illegible], and in what way things will be different. It is hoped that these statements cover all eventualities.

1. There will be no more formal class lectures.
2. Each of you will be provided with what will be called a set of course objectives. These objectives will be presented in four parts.
3. The course objectives will describe in behavioral terms what you are expected to be able to do.
4. The course will be divided into four parts, hereafter called [illegible] "mini-courses".
5. For each mini-course there will be three "performance measures" available. In classical terms a performance measure is a test.

(6) At that time at which you feel that you are ready to demonstrate a proficiency in mini-course k, (k = 1, 4), you will be given an opportunity to demonstrate this by achieving a 90 percent performance level on a randomly selected performance measure.

(7) The opportunities to accomplish item (6) will be during the regularly scheduled class hours [illegible]. Some additional time will be made available, if it is possible to arrange it.

(8) If you achieve 90 percent or better, you will be able to go on to mini-course k+1. If k = 4, you have finished Adv. Calc. I. If you do not achieve 90 percent or better you will have 3 - (no. of times you have already tried) opportunities to demonstrate your proficiency.

(9) If after three tries, you have not achieved 90 percent on a performance measure, a grade of F will be recorded.

(10) I will be available during the regularly scheduled class hours in the assigned classrooms to administer performance measures and to give individual, and group discussion of the subject.

(11) During these periods, I will refrain from discussing politics, the Mid-East situation and philosophizing about anything. I will direct my discussion to the topics in AC I only. I would ask that you cooperate and keep your questions on the subject. We will relegate our discussion of non-essential material to a no credit elective ES 101, to be offered between Christmas and NewYears.

12) In order to supplement the text, I will make available some study guides for each mini-course. A study guide can take one of several possible forms:

(1) A reference to another text. Chapter and page will be cited. Text will be put on "reserve" in the library.
(2) A mimeod direction-sheet.
(3) Others as they become available.

13) There will be no final exam, as such. A 2-hour period will be scheduled during finals week which is to be used in the same way other regularly scheduled classes are; it is expected that anyone not finished with mini-course four will do so at that time.

14) If you have not finished mini-course four by the time of the final exam period, a contract between you and the instructor must be reached. This contract will specifically state a date by which you will finish. Procrastination is not to be fostered.

(15)The grade you receive is to be determined in the following way:

Let x S_{kj} = score in % an mini-course k, attempt an performance measure j.

$k = 1, \dots 4$
$j = 1, \dots 3$

If $S_{pq} \geq 90\%$ then $S_{pj} = 0$ for $j = q + 1, \dots 3$

Let $\mu_{\beta}(x) = \begin{cases} \beta, & x \geq 0 \\ 0, & x < 0 \end{cases}$

Define

$$S = \frac{\sum_{k=1}^{4} \sum_{j=1}^{3} S_{kj}\, \mu_{100}\text{[redacted]}(j-5)\, (S_{kj} - 90)}{400}$$

If $90 \leq S \leq 100$, grade A

$85 \leq S \leq 90$ grade B+

$80 \leq S \leq 85$ grade B

Announcement No. 2, p. 1/AC I/10/22/70/DFO

(1) Line 4, p.2, of Announcement No. 1 should be corrected to read as follows:

$$S = \frac{\sum_{k=1}^{4} \sum_{j=1}^{3} s_{kj}\, \mu_{100} - (j-3)\,(S_{kj} - 90)}{400}$$

(2) Three additional hours will be provided as follows:

(i) W/3/F208

(ii) TE/9/A204

COURSE OBJECTIVES FOR MC_1-907.

Material: Chapters I and II in Text.

If you received a raw score of $\geq$ 80 on the quiz, $S_{11} = 100$, by definition, and you may proceed to MC_2.

Peformance Required for MC_1

(1) To write in words and appropriate symbols any definition in Chapters I and II.

(2) To be able to write a proof, complete with reasons for each step taken, any of the theorems listed below, given the statement of the theorem.

THM I, p. 19	Thm IV, P. 46
Thm V, p. 29	Thm I, p. 48
Thm VII, p. 30	Thm III, p. 48
Thm XIV, p. 32	Thm IV, p. 48
Thm V, p. 38	

(3) Given $f(x)$, a, L, to write an expression for S in terms of ε, and a which demonstrates the validity of the statement

$$\lim_{x \to a} f(x) = L.$$

(4) Given a sequence $\{a_n\}$, L, to write an expression for N in terms of ε which demonstrates the validity of the statement

$$\lim_{n \to \infty} a_n = L$$

(5) In either (3) or (4) a (and, or) L may be one of the following ∞, $-\infty$, $+\infty$. In which case the validity of the appropriate limit statement is to be demonstrated.

(6) To demonstrate the validity of a sequence of statements depending on the positive integers, using the principle of Mathematical Induction, provided;you are given the statements to be validated.

(7) To write a valid proof given either of the statements given in problems 13 and 14, p. 21.

(8) To write an example of a function described in problems 15, 16, 17, p. 51 and to give written verification of the correctness of your example.

The performance measures for MC_1 will be available by 10/22/70.

Study Guide [illegible]

Topic: To apply the definition of a limit & thereby demonstrate the validity of the statement $\lim_{x \to a} f(x) = L$, in terms of the definition.

Problem: Show that $\lim_{x \to 2} \left(\frac{x^2 - 3x}{x-1} \right) = -2$, using the definition of the limit.

Ref: Defn of $\lim_{x \to a} f(x) = L$ is the following:

If $\forall$ $\varepsilon > 0$, $\exists$ $\delta > 0$, $\ni \left(0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon \right)$, then we say $\lim_{x \to a} f(x) = L$

Application to above problem:

(a) Form the difference $|f(x) - L|$

$$\left| \frac{x^2 - 3x}{x-1} - (-2) \right| = \left| \frac{x^2 - 3x + 2(x-1)}{x-1} \right|$$

$$= \left| \frac{x^2 - x - 2}{x-1} \right|$$

(b) Factor, if possible, $(x-a)$ out of the numerator;

$$\left| \frac{x^2 - 3x}{x-1} + 2 \right| = \left| \frac{(x-2)(x+1)}{x-1} \right| = |x-2| \left| \frac{(x+1)}{x-1} \right|$$

3) For the term remaining, besides $x-2$, obtain an upper bound on it. Namely obtain a bound on $\left|\frac{x+1}{x-1}\right|$: so let $\delta_1 = 1/2$ (ie $|x-2| < 1/2$).

We do this to obtain some bound on $\left|\frac{x+1}{x-1}\right|$. $\therefore$ δ is at least, always as small as $1/2$. Now $|x-2| < 1/2 \Rightarrow$

$-1/2 < x-2 < 1/2 \Rightarrow 3/2 < x < 5/2 \Rightarrow$

(i) $5/2 < x+1 < 7/2$ and

(ii) $1/2 < x-1 < 3/2 \Rightarrow \frac{1}{x-1} < 2$

(4) Since for $|x-2| < 1/2$ $x+1 > 0$ & $x-1 > 0$, then using (i)(ii) we have for $|x-2| < 1/2$

$$\frac{x+1}{x-1} = \left|\frac{x+1}{x-1}\right| < \left(\frac{7}{2}\right)(2) = 7$$

(5) Using the result of (4) in (2) we have

$$|x-2| < 1/2 \Rightarrow |x-2|\left|\frac{x+1}{x-1}\right| < 7|x-2|$$

(c) Consequently

$$|x-2| < \tfrac{1}{2} \implies \left| \frac{x^2-3x}{x-1} - (-2) \right| < 7|x-2| < \varepsilon,$$

if $|x-2| < \frac{\varepsilon}{7}$.

$\therefore \left| \frac{x^2-3x}{x-1} - (-2) \right| < \varepsilon$ if $|x-2| < \frac{1}{2}$ or $\frac{\varepsilon}{7}$,

whichever is smaller.

$\therefore$ given ε, we have found that

$$\delta = \min\left(\tfrac{1}{2}, \tfrac{\varepsilon}{7}\right) \implies$$

$$|x-2| < \delta \implies \left| \frac{x^2-3x}{x-1} - (-2) \right| < \varepsilon.$$

$\therefore$ by the definition

$$\lim_{x \to 2} \left(\frac{x^2-3x}{x-1} \right) = -2.$$

LIMIT DEFINITIONS:

$\lim_{n \to +\infty} a_n = a$ IFF $\forall\ \varepsilon > 0\ \exists\ N = N(\varepsilon) \ni n > N \Rightarrow |a_n - a| < \varepsilon$

$\lim_{n \to +\infty} a_n = +\infty$ IFF $\forall\ B\ \exists\ N = N(B) \ni n > N \Rightarrow a_n > B$

$\lim_{n \to +\infty} a_n = -\infty$ IFF $\forall\ B\ \exists\ N = N(B) \ni n > N \Rightarrow a_n < B$

$\lim_{n \to +\infty} a_n = \infty$ IFF $\lim_{n \to \infty} |a_n| = +\infty$

$\lim_{x \to a} f(x) = L$ IFF $\forall\ \varepsilon > 0\ \exists\ \delta = \delta(\varepsilon) > 0 \ni 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$\lim_{x \to +\infty} f(x) = L$ IFF $\forall\ \varepsilon > 0\ \exists\ N = N(\varepsilon) \ni x > N \Rightarrow |f(x) - L| < \varepsilon$

$\lim_{x \to -\infty} f(x) = L$ IFF $\forall\ \varepsilon > 0\ \exists\ N = N(\varepsilon) \ni x < N \Rightarrow |f(x) - L| < \varepsilon$

$\lim_{x \to a} f(x) = +\infty$ IFF $\forall\ B\ \exists\ \delta = \delta(B) > 0 \ni 0 < |x - a| < \delta \Rightarrow f(x) > B$

$\lim_{x \to a} f(x) = -\infty$ IFF $\forall\ B\ \exists\ \delta = \delta(B) > 0 \ni 0 < |x - a| < \delta \Rightarrow f(x) < B$

$f(x)$ IS CONTINUOUS AT $x = a$ IFF

1) IT IS DEFINED AT $x = a$

2) $\forall\ \varepsilon > 0\ \exists\ \delta = \delta(\varepsilon) > 0 \ni |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

Theorems

(Pg 19) IF $\{a,b\} > 0 \ \exists \ n \in P.I \ni na > b$

PROOF: ASSUME $na \leq b \ \forall n$

$\therefore$ $a, 2a, 3a, \ldots$ WOULD BE BOUNDED

LET $c =$ L.U.B. OF THIS SET

$\Rightarrow na \leq c \ \forall n \Rightarrow (n+1)a \leq c \ \forall n$

$\therefore na + a \leq c \Rightarrow na \leq c - a \ \forall n$

HOWEVER $c - a$ IS AN UPPER BOUND $<$ L.U.B

$\therefore$ $na > b$ Q.E.D.

(Pg 29) ANY CONVERGENT SEQUENCE IS BOUNDED

PROOF: ASSUME $a_n \to a$; CHOOSE A DEFINITE NEIGHBORHOOD, SAY $(a-1, a+1)$. IN THAT THIS NEIGHBORHOOD CONTAINS ALL BUT A FINITE NUMBER OF $\{a_n\}$ TERMS, A SUITABLE ENLARGEMENT WILL CONTAIN THEM ALL. Q.E.D.

(Pg 30) $\lim\limits_{n \to +\infty} (a_n - b_n) = \lim\limits_{n \to +\infty} a_n - \lim\limits_{n \to +\infty} b_n$

PROOF: Let $a_n \to a$ AND $b_n \to b$ AND $\varepsilon > 0$

CHOOSE N LARGE ENOUGH SO:

$$|a_n - a| < \tfrac{1}{2}\varepsilon \ ; \ |b_n - b| < \tfrac{1}{2}\varepsilon$$

BY TRIANGE INEQUALITY, FOR $n > N$

$$|(a_n - b_n) - (a - b)| = |(a_n - a) - (b_n - b)|$$
$$\leq |a_n - a| + |b - b_n| < \varepsilon$$ Q.E.D.

(Pg 32) ANY BOUNDED

Pg 21   10-28-70

13) a) IF $S \neq \phi$, S IS BOUNDED ABOVE, THEN $\forall\ s \in S \Rightarrow s < x + \epsilon$

WHERE x IS L.U.B.

PROOF: ASSUME $\exists\ s \in S \ni s \geq x + \epsilon$; $\epsilon > 0$

$\therefore\ s - \epsilon \geq x$

ALSO $s \geq s - \epsilon$

$\Rightarrow\ x \leq s - \epsilon < s \Rightarrow s > x$, COUNTERDICTING THE DEFINITION OF L.U.B.

$\therefore\ \forall\ s \in S \Rightarrow s < x + \epsilon$

b) IF $S \neq \phi$, $\epsilon > 0$, x IS L.U.B OF S, THEN $\exists\ s \in S \Rightarrow s > x - \epsilon$

~~$\forall\ s \in S \Rightarrow s \leq x - \epsilon < x$~~

$f(x)$ $g(x)$

$f[g(x)]$

THEM

IF $f(x)$ IS CONT. ON $[a,b]$ $\exists$ $B \ni$ $a \le x \le b \Rightarrow |f(x)| \le B$

PROOF: ASSUME $f(x)$ CONT, BUT NOT BOUNDED

$\therefore$ $\forall\ n > 0\ \exists\ x_n \in [a,b] \ni |f(x)| > n$

EVERY BOUNDED SEQ. CONTAINS CONV. SUBSEQ

$\therefore$ LET $\{X_{n_k}\} \subset \{X_n\}$ AND $\{X_{n_k}\} \Rightarrow X_0$

$a \le x_{n_k} \le b \Rightarrow a \le x_0 \le b \Rightarrow x_0 \in [a,b]$

$f(x)$ IS CONT AT X_0, $f(X_{n_k}) \to f(X_0)$

BUT $\{f\{X_{n_k}\}\}$ IS BOUNDED

CONTRADICTING $|f(x_{n_k})| > n_K$

Theorems and proofs

THEOREM I (Pg 19) If a and b are positive numbers, there is a positive integer n, such that $na > b$

IF $(a,b) > 0 \ \exists\ n \in P.I \ni na > b$

PROOF: ASSUME $na \leq b \ \forall\ n$

THEN $a, 2a, 3a \ldots$ would be bounded above

LET $C = L.U.B \Rightarrow na \leq c \ \forall\ n$

$\therefore (n+1)a \leq c \ \forall n$

$\Rightarrow na + a \leq c \Rightarrow na \leq c - a \ \forall\ n$

HOWEVER $c - a < c$, THE L.U.B.

This is the desired counterdiction, for $c-a$ is an upper bound less than L.U.B. C

THEOROM V (Pg 29) If c_n converges, it is bounded

ie) $\exists\ P > 0 \ni |C_n| \leq P \ \forall\ n$

PROOF: Let $\lim\limits_{n\to\infty} a_n = a$

Let $\varepsilon = 1 \therefore \ \exists\ N \ni |a_n - a| < 1 \ \forall\ n \leq N$

$\therefore a_n - 1 < a_n < a_n + 1 \ \ \forall\ n \geq N$

This neighborhood conains all but a finite number of terms of $\{a_n\}$, a suitable enlargement will contain these missing terms as well.

THEOREM VIII (Pg 30) $\lim\limits_{n\to\infty} a_n b_n = \lim\limits_{n\to\infty} a_n \lim\limits_{n\to\infty} b_n$

PROOF: Let $a_n \to a$ and $b_n \to b \Rightarrow a_n b_n - ab \to 0$

$a_n b_n - ab = (a_n - a) b_n + a(b_n - b)$

$\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n$

$\Rightarrow (a_n - a) b_n + a(b_n - b) \to 0$

THEOREM I (Pg 48) If $f(x)$ is continuous on $[a, b]$

$\exists B \ni a \leq x \leq b \Rightarrow |f(x)| \leq B$

PROOF: Let $f(x)$ be cont. on $[a, b]$, assume its unbounded

$\therefore \forall n > 0 \ni x_n \in [a, b] \ni |f(x_n)| > n$

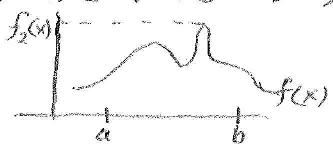
$\{x_{n_K}\} \subseteq \{x_n\}$; $\{x_{n_K}\}$ converges to x_0 ie $x_{n_K} \rightarrow x_0$

$a \leq x_{n_K} \leq b \Rightarrow a \leq x_0 \leq b \therefore x_0 \in [a, b]$

Since $f(x)$ is cont. at x_0, $f(x_{n_K}) \rightarrow f(x_0) \Rightarrow \{f(x_{n_K})\}$ is bounded, contradicting $|f(x_n)| > n_K$

THEOREM II (Pg 48) If $f(x)$ is cont. on $[a, b]$ $\exists x_1 \,\&\, x_2 \in [a, b] \ni a \leq x \leq b \Rightarrow f(x_1) \leq f(x) \leq f(x_2)$

PROOF:



Let $f(x)$ be continuous on $[a, b]$ $\Rightarrow f(x)$ is bounded $\Rightarrow$ has L.U.B. $= M$

Let $x_n \in [a, b] \ni |f(x_n) - M| < \frac{1}{n} \Rightarrow \lim_{n \to +\infty} f(x_n) = M$

since $\{x_n\}$ is bounded $\exists \{x_{n_K}\}$ conv. subsequence of $\{x_{n_K}\}$

let $\{x_{n_K}\} \rightarrow x_0$. $\because f(x)$ cont. at x_0, $f(x_{n_K}) \Rightarrow f\{x_0\}$

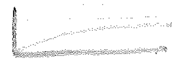
$f\{x_n) \rightarrow M$ and $f\{x_{n_K}) \rightarrow M$

Lim is unique $\Rightarrow f(x_0) = M$

THEOREM III (Pg 48) If $f(x) \subseteq [a, b]$, and if $a \,\&\, b$ have opposite signs, $\exists x_0 \ni f(x_0) = 0$

Pg 28-9

20-26) $A = \left\{ \frac{2n+1}{2n} \;\middle|\; n = 1, 2, 3, \ldots \right\}$



$\lim_{n\to\infty} \frac{2n+1}{2n} = 1$

$\lim_{n\to\infty} a_n = L$ IFF $\forall\ \varepsilon > 0\ \exists\ N(\varepsilon) \ni n > N(\varepsilon) \Rightarrow |a_n - a| < \varepsilon$

$\left| \frac{2n+1}{2n} - 1 \right| < \varepsilon$

$\frac{1}{2n} < \varepsilon \Rightarrow 1 < 2n\varepsilon \Rightarrow \frac{1}{2\varepsilon} < n$

B+ N(B)

22-26) $\lim_{n\to\infty} n^2 = +\infty$

$\lim_{n\to\infty} a_n = +\infty$ IFF $\forall B\ \exists\ N(B) \ni n > N \Rightarrow a_n > B$

Let $N(B) = B^2$

Bob Marks

first

Pg 11 9-15-70

13) ESTABLISH: $1^2+2^2+\cdots n^2 = \frac{1}{6}n(n+1)(2n+1) = f(n)$

1) $f(1) \Rightarrow 1^2 = \frac{1}{6}\,1\,(1+1)(2+1)$

$1 = 1$

2) $1^2+2^2+\ldots+(n+1)^2 = \frac{1}{6}(n+1)(n+2)\left[2(n+1)+1\right]$
$= \frac{1}{6}(n+1)(n+2)(2n+3)$

3) $\frac{1}{6}(n+1)(n+2)(2n+3) = \frac{1}{6}n(n+1)(2n+1) + (n+1)^2$
$= (n+1)\left[\frac{n}{6}(2n+1)+n+1\right]$
$= (n+1)\left[\frac{n^2}{3}+\frac{n}{6}+n+1\right]$

$(n+1)(n+2)(2n+3) = (n+1)(2n^2+7n+6)$

$2n^2+3n+4n+6 = 2n^2+7n+6$

$2n^2+7n+6 \equiv 2n^2+7n+6$

14) ESTABLISH: $1^3+2^3+\ldots+n^3 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$

1) $f(1) \Rightarrow 1 = \frac{1}{30}(2)(3)(5) = 1$

2) $1^3+2^3+\ldots+(n+1)^3 = \frac{1}{30}(n+1)(n+2)(2n+3)\left[(n+1)^2 3 + 3n+2\right]$

3) $\frac{1}{30}(n+1)(n+2)(2n+3)\left[3n^2+6n+6+3n+2\right] =$
$= \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1) + (n+1)^3$

$\frac{1}{30}(n+2)(2n+3)(3n^2+9n+8) = \frac{1}{30}n(2n+1)(3n^2+3n-1) + (n+1)^2$

$(n+2)(2n+3)(3n^2+9n+8) = n(2n+1)(3n^2+3n-1) + 30(n+1)^2$

$(2n^2+7n+6)(3n^2+9n+8) = (2n^2+n)(3n^2+3n-1) + (30n^2+60n+30)$

$6n^4+39n^3+97n^2+110n+48 = 6n^4+9n^3+n^2$

NUTZ!

(OVER)

Pg 20 9-17-70

1) a) T.B.P: $\forall r \in R \neq 0$ & $\forall x \in IRR$, $(x \pm r = T) \in IRR$

ASSUME: $T \in R$

$x \pm r = T$

$x = T \mp r$

$T \in R$ & $r \in R \Rightarrow T$ & R MAY BE REPRESENTED BY INTEGER RATIOS

LET $T = \frac{P_1}{Q_1}$ & $R = \frac{P_2}{Q_2}$

$\therefore$ $X = \frac{P_1}{Q_1} \mp \frac{P_2}{Q_2} = \frac{P_1Q_2 \mp P_2Q_1}{Q_1Q_2}$

SUMS AND PRODUCTS OF INT. ARE INT

$\therefore$ LET $\frac{P_1Q_2 \mp P_2Q_1}{Q_1Q_2} = \frac{P_3}{Q_3}$

ERGO $X = P_3/Q_3$ WHERE P_3 & $Q_3 \in$ INTEGERS

$\Rightarrow X \in R$, A COUNTERDICTION TO $X \in IRR$ Q.E.D.

b) T.B.P.: $\forall r \in R \neq 0$ & $\forall x \in IRR$, $(xr) \in IRR$

ASSUME: $xr \in R$; LET $xr = T$ AND $T = \frac{A}{B} \ni (A$ & $B) \in IN.$

$xr = \frac{A}{B} \Rightarrow x = \frac{A}{Br}$

SUMS & PRODUCTS OF INT. ARE INT.

$\therefore$ LET $\frac{A}{Br} = \frac{A}{m} \ni M = Br$ AND $M \in I$

ERGO $X = \frac{A}{m}$

$\Rightarrow X \in R$, A COUNTERDICTION OF $X \in IRR$ QED

c) T.B.P. $\forall r \in R \neq 0$ & $\forall x \in IRR$, $(\frac{x}{r} = A) \in IRR$.

ASSUME: $(\frac{x}{r} = A) \in R$

LET $A = \frac{m}{}$

b) T.B.P : $\forall r \in R \neq 0$ & $\forall x \in IRR$, $(xr = m) \in IRR$

ASSUME $(xr = m) \in R$

LET $m = \frac{A}{B} \ni (A, B) \in IN.$, AND $r = \frac{C}{D} \ni (C, D) \in IN$

$\therefore$ $xr = m \Rightarrow x\frac{C}{D} = \frac{A}{B} \Rightarrow x = \frac{AD}{BD}$

PRODUCTS OF INTEGERS ARE INT

$\therefore$ LET $AD = E$ & $BD = F \ni (E, F) \in INT$

ERGO $X = \frac{AD}{BD} = \frac{E}{F}$

$\Rightarrow X \in R$, A COUNTERDICTION TO $X \in IRR$

QED

Pg 3 9-20-70

1) PROVE UNIQUENESS OF 0

ASSUME $0'$ AND 0 EXIST

$$0'+0=0'+0 \Rightarrow 0=0'$$

2) PROVE CANCELATION IN ADDITION

ie $X+Y=X+Z \Rightarrow Y=Z$

$$-X+(X+Y)= -X+(X+Z)$$

$$(-X+X)+Y=(-X+X)+Z$$

$$\Rightarrow 0+Y=0+Z \Rightarrow Y=Z$$

3) PROVE $X\cdot 0=0$

$$X\cdot 0=X(0+0)=X\cdot 0+X\cdot 0$$

$$\therefore X\cdot 0=X\cdot 0+X\cdot 0$$

$$X\cdot 0+0=X\cdot 0+X\cdot 0$$

$$\Rightarrow 0=X\cdot 0$$

4) PROVE $\forall X\neq 0 \;\&\; \forall Y\neq 0,\ XY\neq 0$

ASSUME $X\neq 0 \ni Y\neq 0 \;\&\; XY=0$

$$XY=0$$

$$X^{-1}(XY)=0 \;;\; (X^{-1}X)Y=0$$

$\Rightarrow Y=0$, A COUNTERDICTION

Pg. 21 9-20-70

13) 

GIVEN SET S WITH L.U.B = X ; $\epsilon > 0$ AND $s \in S$

PROVE $s < x + \epsilon$

($s \leq x$ BY DEF)

ASSUME: $s \geq x + \epsilon$

$s \leq x \Rightarrow x \geq s \geq x + \epsilon$

$\therefore x \geq x + \epsilon$

IN THAT $\epsilon > 0$;

$x > x + \epsilon$

$\therefore x > s > x + \epsilon$

WHICH PUTS s ABOVE L.U.B., WHICH MAKES IT NO LONGER AN ELEMENT OF S; A COUNTER-DICTION TO $s \in S$.

$\therefore s < x + \epsilon$

Pg 28-9 9-22-70

26-20) $\{a_n\} = \frac{2n+1}{2n}$; $\lim_{n\to\infty} a_n = 1$

$$|a_n - 1| = \left|\frac{2n-1}{2n} - 1\right| = \left|\frac{1}{2n}\right| < \varepsilon \ni (\varepsilon > 0)$$

$$\therefore |n| > \frac{1}{2\varepsilon}$$

$$\Rightarrow \text{LET } N(\varepsilon) \in (\text{P.I} > \frac{1}{2\varepsilon})$$

21-27) $\frac{3}{5}, \frac{3}{7}, \frac{5}{9}, \ldots, \frac{n+1}{2n+3}, \frac{n+2}{2n+3}, \ldots$

$$a = \lim_{n\to\infty} \frac{n+1}{2n+3} + \lim \frac{n+2}{2n+3}$$

$$= \lim_{n\to\infty} \frac{2n+3}{2n+3} = 1$$

$|a_n - 1| = 0 \Rightarrow$ SEQUENCE DIVERGES

23-29) $\frac{3}{7}, \frac{-8}{7}, \frac{13}{7}, \frac{-18}{7}, \ldots \frac{(-1)^{n+1}(5n-2)}{7}$

SEQUENCE DIVERGES

NO FINITE INTERVAL ABOUT ANY POINT CAN CONTAIN ALL OF THE TERMS OF THIS SEQUENCE

Pg 73 10-15-70

1) $Y = X^2 - 4X + 7$

$$\frac{dY}{dX} = \lim_{\Delta x \to 0} \frac{(X+\Delta X)^2 - 4(X+\Delta X) + 7 - X^2 + 4X - 7}{\Delta X}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{X^2} + 2\Delta X X + (\Delta X)^2 - \cancel{4X} - 4\Delta X - \cancel{X^2} + \cancel{4X}}{\Delta X}$$

$$= \lim_{\Delta x \to 0} 2X + \Delta X - 4 = 2X - 4$$

2) $Y = X^3$

$$\frac{dY}{dX} = \lim_{\Delta x \to 0} \frac{(X+\Delta X)^3 - X^3}{\Delta X}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{X^3} + 3X^2\Delta X + 3X(\Delta X)^2 + (\Delta X)^3 - \cancel{X^3}}{\Delta X}$$

$$= \lim_{\Delta x \to 0} 3X^2 + 3X\Delta X + (\Delta X)^2$$

$$= 3X^2$$

3) $Y = \frac{1}{X^2}$

$$\frac{dY}{dX} = \lim_{\Delta x \to 0} \frac{\frac{1}{(X+\Delta X)^2} - \frac{1}{X^2}}{\Delta X}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{1}{X^2 + 2X\Delta X + \Delta X^2} - \frac{1}{X^2}}{\Delta X}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{X^2 - X^2 - 2X\Delta X - \Delta X^2}{X^2(X^2 + 2X\Delta X + \Delta X^2)}}{\Delta X}$$

$$= \lim_{\Delta X \to 0} \frac{-2X\Delta X - \Delta X^2}{\Delta X X^2(X^2 + 2X\Delta X + \Delta X^2)}$$

$$= \lim_{\Delta x \to 0} \frac{-2X - \Delta X}{X^4 + 2X^3\Delta X + X^2\Delta X^2}$$

$$= \frac{-2X}{X^4} = \frac{-2}{X^3}$$

4) $Y = \frac{3X+2}{5X-4}$

$$\frac{dY}{dX} = \lim_{\Delta x \to 0} \frac{\frac{3(X+\Delta X)+2}{5(X+\Delta X)-4} - \frac{3X+2}{5X-4}}{\Delta X}$$

$$= \lim_{\Delta x \to 0} \frac{(3X+3\Delta X+2)(5X-4) - (3X+2)(5X+5\Delta X-4)}{\Delta X\,[5X+5\Delta X-4][5X-4]}$$

$$= \lim_{\Delta x \to 0} \frac{(15X^2 + 15X\Delta X + 10X - 12X - 12\Delta X - 8) - (15X^2 + 15X\Delta X - 12X + 10X + 10\Delta X - 8)}{\Delta X\,(25X^2 + 25X\Delta X - 20X - 20X - 20\Delta X + 16)}$$

$$= \lim_{\Delta x \to 0} \frac{-22}{25X^2 + 25X\Delta X - 40X - 20\Delta X + 16}$$

$$= \frac{-22}{25X^2 - 40X + 16}$$

8) $Y = X^n$

$$\frac{dY}{dX} = \lim_{\Delta x \to 0} \frac{(X+\Delta X)^n - X^n}{\Delta X}$$

$$= \lim_{\Delta x \to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} X^{n-k} \Delta X^{k} - X^n}{\Delta X}$$

$$= \lim_{\Delta x \to 0} \sum_{k=1}^{n} \binom{n}{k} X^{n-k} \Delta X^{k} / \Delta X$$

$$= \lim_{\Delta x \to 0} \sum_{k=1}^{n} \binom{n}{k} X^{n-k} \Delta X^{k-1}$$

$$= \lim_{\Delta x \to 0} \sum_{k=0}^{n} \binom{n}{k} X^{n-(k+1)} \Delta X^{k-1}$$

$$= \lim_{\Delta X \to 0} \sum_{k=0}^{n} \binom{n}{k} X^{n-(k-1)} X^{-2} \Delta X^{k-1}$$

① DEFINITIONS

1) $X < Y \Leftrightarrow Y - X$ IS POSITIVE
$X > Y \Leftrightarrow X - Y$ IS POSITIVE
2) X IS NEGATIVE $\Leftrightarrow -X$ IS POSITIVE
3) $X \leq Y \Leftrightarrow (X < Y) \cup (X = Y)$
$X \geq Y \Leftrightarrow (X > Y) \cup (X = Y)$
4) AN INDUCTIVE SET OF NUMBERS HAVE THE FOLLOWING PROP.:
a) THE NUMBER 1 IS A MEMBER OF A $(1 \in A)$
b) $\forall X \in A \; \exists \; (X+1) \in A$
5) $X \in$ R.N. $\Leftrightarrow \exists (p,q) \in I \ni X = \frac{p}{q}$
$X \notin$ R.N. $\Leftrightarrow X \in$ IRR.
6) a) OPEN INTERVAL (a, b), $\Leftrightarrow a < x < b$
b) CLOSED INTERVAL $[a, b] \Leftrightarrow a \leq x \leq b$
c) HALF-OPEN INTERVALS $(a, b] \Leftrightarrow a < x \leq b$
$[b, a) \Leftrightarrow a \leq x < b$
d) a AND b ARE CALLED ENDPOINTS
e) INFINITE INTERVALS $(a, +\infty) \Leftrightarrow x > a$; $(-\infty, a) \Leftrightarrow x < a$
$[a, +\infty) \Leftrightarrow x \geq a)$; $[-\infty, a] \Leftrightarrow x \leq a$
7) ABSOLUTE VALUE $\Leftrightarrow |x| = \begin{cases} x & \text{IF } x \geq 0 \\ -x & \text{IF } x < 0 \end{cases}$
8) EPSILON NEIGHBORHOOD OF A POINT (OPEN INTERVAL)
$(a - \varepsilon, a + \varepsilon)$ WHERE $\varepsilon > 0$
9) LET D & $R \neq \phi$, THEN FUNCTION WITH DOMAIN OF DEFINITION D & RANGE OF VALUES R, IS A SET f OF ORDERED PAIRS ~~∌~~ $(x, y) \ni x \in D$ & $y \in R$, HAVING PROP.:
a) $\forall x \in D$, $\exists$ AT LEAST ONE $y \in R \ni (x, y) \in f$
b) $\forall y \in R$, $\exists$ AT LEAST ONE $x \in D \ni (x, y) \in f$
f IS SINGLE VALUED IFF $\forall (x, y)$ & $(x, z) \in f$, $y = z$
10) IF ~~∀~~ $Y = f(x)$ WITH D & R, THEN D = INDEPENDENT VARIABLE, R = DEPENDENT VARIABLE.
IF R CONTAINS 1 OBJECT, $f(x)$ IS A CONSTANT FUNCTION
IF D CONTAINS ALL R.N., $f(x)$ IS A REAL VARIABLE FUNCTION
IF R CONTAINS ALL R.N, $f(x)$ IS A REAL-VALUED FUNCTION

COURSE OBJECTIVES FOR MC_2 - 9070

Material: Chapter III of text sec. 301-313 included.

Performance required for MC_1

(1) To write in words and appropriate symbols, any definition given in sec301-313.

(2) To write a proof, complete with reasons for each step taken, any of the theorems listed below, given the statement of the Theorem.

Thm I, p. 69 Chain rule.
Thm I, p. 75
Thm II, p. 75 Rolle's Thm
Thm III, p. 76 Mean-Value Theorem for Derivatives.
Thm I, p. 79
Thm II, P. 79
Thm III, p. 79
Thm , p. 81
ThmII, p. 86
Thm III, p. 86

(3) To write an expression for a function which exhibits one of the following properties, given the property which it must exhibit; and to demonstrate that the function you have given, has the property listed.

(i) a: a real number. $f(x) = |x|$; $a = 0$
Property: a function continuous at $x = a$, but not differentiable at $x = a$.

(ii) a: real number $f(x) = x^2 \sin\frac{1}{x}$ $x \neq 0$; $f(x) = 0$ $x = 0$; $a = 0$
Property: a function, differentiable at $x = a$, but whose derivative is discontinuous at $x = a$.

(4) To prove, by Mathematical Induction, Leibnitz's Rule, given the statement of the rule.

(5) Problems 18, 19, 20, 21, 22, 31 on p. 83. Problem 43, p. 84.

(6) To write a statement of each of the following:

(1) First derivative test.
(2) Second derivative Test.

(7) Given $f(x)$ and an interval $[a,b]$, to find, using the first and second derivative tests, all max and min of $f(x)$ in $[a,b]$.

(8) Problem 9 p. 88, Problem 22 p. 90.

(9) Problems 11, 12, 13, 16, 17, 23, 24, 28, 29 p. 94.

MINI-COURSE #2
THEOREMS:

I) THE CHAIN RULE: IF Y IS A DIFFERENTIABLE FUNCTION OF U AND U IS A DIFFERENTIABLE FUNCTION OF X, THEN Y, AS A FUNCTION OF X, IS DIFFERENTIABLE, AND

$$\frac{dY}{dx} = \frac{dY}{dU}\frac{dU}{dX}$$

PROOF: LET $U \equiv f(x)$ BE DIFF. AT $X = X_0$

$Y \equiv g(x)$ BE DIFF. AT $U = U_0 = f(X_0)$

$h(x) = g[f(x)]$

$$U_0 + \Delta U = f(X_0 + \Delta X)$$

$$\Rightarrow \Delta U = f(X_0 + \Delta X) - f(X_0)$$

$$\text{ALSO } \Delta Y + Y = g(U + \Delta U)$$

$$\Rightarrow \Delta Y = g(U + \Delta U) - g(U)$$

$$\Rightarrow \Delta Y = g(U_0 + \Delta U) - g(U_0)$$

$$\Rightarrow \Delta Y = g[f(X_0 + \Delta X)] - g[f(X_0)]$$

$$\Rightarrow \Delta Y = h(X_0 + \Delta X) + h(X_0)$$

LET $\mathcal{E}(\Delta U) = \frac{\Delta Y}{\Delta U} - \frac{dY}{dU}$

$$\Delta U + U \equiv f(X + \Delta X)$$

$$\lim_{\Delta U \to 0}(U + \Delta U) = \lim_{\Delta X \to 0} f(X + \Delta X)$$

$$U \equiv f(X)$$

$$\Rightarrow (\Delta U \to 0) \Leftrightarrow (\Delta X \to 0)$$

$$\therefore \lim_{\Delta X \to 0} \mathcal{E}(\Delta X) = \lim_{\Delta U \to 0} \mathcal{E}(\Delta X) = \lim_{\Delta X \to 0}\left[\frac{\Delta Y}{\Delta X} - \frac{dY}{dX}\right]$$

$$= \frac{dY}{dU} - \frac{dY}{dU} = 0$$ [LIMIT OF DIFFERENCES = DIFFER. OF LIMITS]

$$\because Y + \Delta Y \equiv g(U + \Delta U)$$

$$\lim_{\Delta Y \to 0}(Y + \Delta Y) = \lim_{\Delta U \to 0} g(U + \Delta U)$$

$$Y \equiv g(U) \Rightarrow (\Delta Y \to 0) \Leftrightarrow (\Delta U \to 0) \Leftrightarrow (\Delta X \to 0)$$

$$\Delta Y = \frac{dY}{dU}\Delta U - \mathcal{E}(\Delta U) \cdot \Delta U$$

$$\frac{\Delta Y}{\Delta X} = \frac{dY}{dU}\frac{\Delta U}{\Delta X} - \frac{\Delta U}{\Delta X}\mathcal{E}(\Delta U)$$

$$\lim_{X \to 0}\frac{\Delta Y}{\Delta X} = \lim_{X \to 0}\left[\frac{dY}{dU}\frac{\Delta U}{\Delta X} - \frac{\Delta U}{\Delta X}\underbrace{\mathcal{E}\,\Delta U}_{=0}\right]$$

$$= \lim_{X \to 0}\frac{dY}{dU}\frac{\Delta U}{\Delta X}$$

$$\Rightarrow \frac{dY}{dX} = \frac{dY}{dU}\frac{dU}{dX}$$

II) LEIBNITZ'S RULE: IF u & v ARE FUNCTIONS OF x, EACH OF WHICH POSSESSES DERIVITIVES OF ORDER n, THEN THE PRODUCT ALSO DOES AND:

$$\frac{d^n}{dx^n}(uv) = \frac{d^n}{dx^n}v + \binom{n}{1}\frac{d^{n-1}v}{dx^{n-1}}\frac{du}{dx} + \binom{n}{2}\frac{d^{n-2}v}{dx^{n-2}}\frac{d^2u}{dx^2} + \dots + u\frac{d^nv}{dx^n}$$

PROOF: BY MATHEMATICAL INDUCTION

1) FOR $n=1$:

$$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\Delta[u(x)\cdot v(x)] = u(x+\Delta x)\,v(x+\Delta x) - u(x)\,v(x)$$

$$\because \left[\begin{array}{l} f(x) = y \\ f(x+\Delta x) = y + \Delta y \\ \Delta y = f(x+\Delta x) - f(x) \end{array}\right]$$

$$\Delta[u(x)\cdot v(x)] = u(x+\Delta x)\,v(x+\Delta x) - u(x+\Delta x)\,v(x)$$
$$+\, u(x+\Delta x)\,v(x) - u(x)\,v(x)$$
$$= u(x+\Delta x)\left[v(x+\Delta x) - v(x)\right] + v(x)\left[u(x+\Delta x) - u(x)\right]$$

$$\frac{\Delta[u\cdot v]}{\Delta x} = u(x+\Delta x)\left[\frac{v(x+\Delta x) - v(x)}{\Delta x}\right] + v(x)\left[\frac{u(x+\Delta x) - u(x)}{\Delta x}\right]$$

$$\lim \frac{\Delta[uv]}{\Delta x} = \frac{d(uv)}{dx} = u(x)\frac{dv(x)}{dx} + v(x)\frac{du(x)}{dx}$$

2) ASSUME FOR:

$$\frac{d^{n-1}(uv)}{dx^{n-1}} = \frac{d^{n-1}u}{dx^{n-1}}\cdot v + \binom{n-1}{1}\frac{d^{n-2}u}{dx^{n-2}}\frac{dv}{dx} + \binom{n-1}{2}\frac{d^{n-3}u}{dx^{n-3}}\frac{d^2v}{dx} +$$
$$\dots\ u\cdot\frac{d^{n-1}v}{dx^{n-1}}$$

$$\frac{d}{dx}\left[\frac{d^{n-1}(uv)}{dx^{n-1}}\right] = \frac{d^n(uv)}{dx} = \frac{d^nu}{dx^n}\cdot v + \frac{d^{n-1}u}{dx^{n-1}}\cdot\frac{dv}{dx}$$
$$+ \binom{n-1}{1}\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + \binom{n-1}{1}\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2v}{dx^2} + \binom{n-1}{2}\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2v}{dx^2}$$
$$+ \dots + u\frac{d^nv}{dx^n}$$

$$\frac{d^n(uv)}{dx} = \frac{d^nu}{dx^n}v + \frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx}\left[1 + \binom{n-1}{1}\right] + \frac{d^{n-2}u}{dx^{n-2}}\frac{d^2v}{dx^2}\left[\binom{n-1}{1} + \binom{n-2}{2}\right]$$
$$+ \frac{d^{n-3}u}{dx^{n-3}}\frac{d^3v}{dx^3}\left[\binom{n-1}{2} + \binom{n-1}{3}\right] + \dots\ u\frac{d^nv}{dx^n}$$
$$= \frac{d^nu}{dx^n}v + \binom{n}{1}\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + \binom{n}{2}\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2v}{dx^2} + \dots + u\frac{d^nv}{dx^n}$$

III) THEOREM: IF $f(x)$ IS CONT. ON $[a,b]$, AND DIFFERENTIABLE ON (a,b), AND $f(x)$ ASSUMES EITHER A MAX OR MIN ON (a,b) $\exists\, \mathcal{E} \ni f'(\mathcal{E})=0$

PROOF: ASSUME $f(\mathcal{E}) = MAX$

$$a < \mathcal{E} < b$$

LET $\dfrac{\Delta Y}{\Delta X} = \dfrac{f(\mathcal{E}+\Delta X) - f(\mathcal{E})}{\Delta X}$

CHOOSE $\Delta X \ni (\mathcal{E}+\Delta X) \in (a,b)$

$f(\mathcal{E}) \geq f(\mathcal{E}+\Delta X) \because f(\mathcal{E})$ IS THE MAX.

$$\Rightarrow \Delta Y \leq 0$$

$\therefore \dfrac{\Delta Y}{\Delta X} \geq 0 \quad \forall \quad \Delta X < 0$

$\dfrac{\Delta Y}{\Delta X} \leq 0 \quad \forall \quad \Delta X > 0$

ASSUMING $\lim\limits_{x \to 0^+} \dfrac{\Delta Y}{\Delta X} = \lim\limits_{x \to 0^-} \dfrac{\Delta Y}{\Delta X}$

$$0 \leq \lim_{x \to 0} \frac{\Delta Y}{\Delta X} \leq 0 \Rightarrow \lim_{\Delta x \to 0} \frac{\Delta Y}{\Delta X} = \frac{dY}{dX} = 0$$

IV) ROLLES THEOREM: IF $f(x)$ IS CONT. ON $[a,b]$, IF $f(a)=f(b)=0$, AND IF $f(x)$ IS DIFFERINTIABLE ON (a,b), $\exists\, \mathcal{E} \in (a,b) \ni f'(\mathcal{E}) = 0$

PROOF: A FUNCTION CONTINUOUS ON $[a,b]$ ON A CLOSED INTERVAL HAS A MAX AND MIN THERE

CASE 1: $MAX = MIN = 0 \Leftrightarrow f(x) = 0 \Rightarrow f'(x) = 0 \Leftrightarrow \mathcal{E} = \{x \mid x \in [a,b]\}$

CASE 2: $\exists\, x_0 \in [a,b] \ni f(x_0) > 0 \Rightarrow f(a) = f(b) \neq MAX$

$\therefore MAX \in (a,b)$

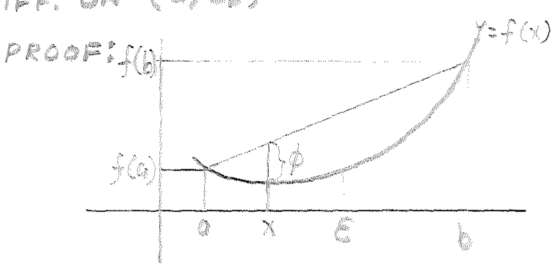
CASE 3: $\exists\, x_0 \in [a,b] \ni f(x_0) < 0 \Rightarrow f(a) = f(b) \neq MIN$

$\therefore MIN \in (a,b)$

FOR CASES 2 & 3: IF $f(x)$ IS CONT. ON $[a,b]$ AND DIFF. ON (a,b), AND $f(x)$ ASSUMES A MAX OR MIN ON (a,b), $\exists\, \mathcal{E} \ni f'(\mathcal{E}) = 0$

V) LAW OF THE MEAN: IF $f(x)$ IS CONT. ON $[a,b]$ AND DIFF. ON (a,b), THEN $\exists\, \mathcal{E} \in (a,b) \ni f'(\mathcal{E}) = \dfrac{f(b)-f(a)}{b-a}$

PROOF:



LET $K = \dfrac{f(b)-f(a)}{b-a} \Rightarrow f(b) = f(a) - K(b-a)$

$Y_1 = f(b) - K(b-x)$ IS LINE THRU $(b, f(b))$ WITH SLOPE K

$Y = f(x)$

LET $\phi(x) = Y_1 - Y$

$$= f(b) - f(x) - K(b-x)$$

$$\phi(a) = f(b) - f(a) - K(b-a) = f(b) - f(a) - [f(b) - f(a)] = 0$$

$$\phi(b) = f(b) - f(b) - K(b-b) = 0$$

$\therefore$ BY ROLLES THEOREM, $\exists\, \mathcal{E} \in (a,b) \ni \phi'(\mathcal{E}) = 0$

$$\phi(\mathcal{E}) = f(b) - f(\mathcal{E}) - Kb + K\mathcal{E}$$

$$\phi'(\mathcal{E}) = -f'(\mathcal{E}) + K = 0 \Rightarrow f'(\mathcal{E}) = K = \frac{f(b)-f(a)}{b-a}$$

VI.) A FUNCTION WITH AN IDENTICALLY VANISHING DERIVITIVE THROUGHOUT AN INTERVAL MUST BE CONSTANT IN THAT INTERVAL. i.e.)

$$f'(x)=0 \Rightarrow f(x) = \text{CONSTANT}$$

PROOF: ASSUME $f'(x)=0 \Rightarrow f(x) = $ NON-CONSTANT

$\therefore \exists\, [a \neq b] \in$ INTERVAL $\ni f(a) \neq f(b)$

BY LAW OF THE MEAN: $\exists\, \mathcal{E} \in (a,b) \ni$

$$f'(\mathcal{E}) = \frac{f(b)-f(a)}{b-a} \neq 0$$

IN COUNTERDICTION TO ASSUMPTION

VII) $g'(x) = f'(x) \Rightarrow g(x) = f(x) + \text{CONST.}$

PROOF: LET $h'(x) = g'(x) - f'(x) = 0$

THEN $h(x) = g(x) - f(x) = \text{CONST.}$

$\Rightarrow g(x) = f(x) + \text{CONST}$

VIII) IF $f(x)$ IS CONTINUOUS OVER AND INTERVAL, AND DIFFERENTIABLE OVER THE INTERVAL:

1) $f'(x) \geq 0 \Rightarrow f(x)$ IS MONO. INCR.
2) $f'(x) \leq 0 \Rightarrow f(x)$ IS MONO DECR.
3) $f'(x) > 0 \Rightarrow f(x)$ IS STRICTLY INCR.
4) $f'(x) < 0 \Rightarrow f(x)$ IS STRICTLY DECR.

PROOF: LET $x_1, x_2 \in$ INTERVAL. BY LAW OF THE MEAN, $\exists\, x_3 \ni$

$$f'(x_3) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

W.L.O.G., LET $x_2 > x_1 \Rightarrow x_2 - x_1 > 0$

1) $f'(x_3) \geq 0 \Rightarrow f(x_2) - f(x_1) \geq 0 \Rightarrow f(x)$ IS MONO. INCR
2) $f'(x_3) \leq 0 \Rightarrow f(x_2) - f(x_1) \leq 0 \Rightarrow f(x)$ IS MONO. DECR.
3) $f'(x_3) > 0 \Rightarrow f(x_2) - f(x_1) > 0 \Rightarrow f(x)$ IS STRICTLY INCR
4) $f'(x_3) < 0 \Rightarrow f(x_2) - f(x_1) < 0 \Rightarrow f(x)$ IS STRICTLY DECR.

IX) EXTENDED LAW OF THE MEAN:

IF $f(x), f'(x), \ldots f^{n-1}(x)$ ARE CONT ON $[a,b]$ AND IF $f^{n}(x)$ EXISTS ON (a,b), THEN $\exists\ \mathcal{E} \in (a,b) \ni$

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \ldots + \frac{f^{n-1}(a)}{(n-1)!}(b-a)^{n-1} + \frac{f^{n}(\mathcal{E})}{n!}(b-a)^n$$

PROOF:

LET K BE DEFINED BY:

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \ldots + \frac{f^{n-1}(a)}{(n-1)!}(b-a)^{n-1} + \frac{K}{n!}(b-a)^n$$

LET $\phi(x) = f(b) - f(x) - f'(x)(b-x) - \frac{f''(x)}{2!}(b-x)^2 \ldots - \frac{f^{n-1}x}{(n-1)!}(b-x)^{n-1} - \frac{K}{n!}(b-x)^n$

$$\phi(a) = f(b) - f(a) - f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 \ldots - \frac{f^{n-1}a}{(n-1)!}(b-a)^{n-1} - \frac{K}{n!}(b-a)^n$$
$$= f(b) - f(b) = 0$$

$$\phi(b) = f(b) - f(b) - f'(x)(b-b) - \frac{f''(b)}{2!}(b-b)^2 \ldots = 0$$

$\therefore$ BY ROLLES THEM. $\exists\ \mathcal{E} \in (a,b) \ni \phi'(\mathcal{E}) = 0$

$$0 = \phi'(x) = -f'(x) + f'(x) - f''(x)(b-x) - \ldots - \frac{f^{n}x}{(n-1)!}(b-x)^{n-1} + \frac{K}{(n-1)!}(b-x)^{n-1}$$

$$\Rightarrow K = f^{n}(\mathcal{E})$$

X) FIRST DERIVITIVE TEST

IF $f(x)$ IS CONT AT $x = \mathcal{E}$, AND DIFFERENTIABLE IN A DELETED NEIGHBORHOOD OF $\mathcal{E}$, AND IF, IN THIS NEIGHBORHOOD, $f'(x) > 0$ FOR $x < \mathcal{E}$ AND $f'(x) < 0$ FOR $x > \mathcal{E}$ ($f'(x) < 0$ FOR $x < \mathcal{E}$ AND $f'(x) > 0$ FOR $x > \mathcal{E}$), THEN $f(x)$ HAS A RELATIVE MAX (MIN) AT $x = \mathcal{E}$. IF $f'(x)$ HAS A CONSTANT SIGN THROUGHOUT THE DELETED NEIGHBORHOOD, $f(x)$ HAS NEITHER A REL. MAX OR MIN AT $x = \mathcal{E}$.

PROOF: LET X BE AN ARBITRARY PT. IN DELETED NEIGHBORHOOD OF $\mathcal{E}$. BY LAW OF MEAN, $\exists\ \mathcal{E}' \ni$

$$f(x) - f(\mathcal{E}) = f'(\mathcal{E}')(x - \mathcal{E})$$

$f'(x) > 0$ FOR $x < \mathcal{E} \Rightarrow f(x)$ IS STRICTLY INCREASING
$f'(x) < 0$ FOR $x > \mathcal{E} \Rightarrow f(x)$ IS STRICTLY DECREASING
$\therefore f(\mathcal{E}) > f(x) \Rightarrow f(\mathcal{E})$ IS MAX

$f'(x) < 0$ FOR $x < \mathcal{E} \Rightarrow f(x)$ IS STRICTLY DECREASING
$f'(x) > 0$ FOR $x > \mathcal{E} \Rightarrow f(x)$ IS STRICTLY INCREASING
$\therefore f(\mathcal{E}) < f(x) \Rightarrow f(\mathcal{E})$ IS MIN

XI) SECOND DERIVITIVE TEST: IF $f(x)$ IS DIFFERINTIABLE IN A NEIGHBORHOOD OF CRITICAL VALUE $\mathcal{E}$, AND IF $f''(\mathcal{E})$ EXISTS AND IS NEGATIVE (POSITIVE), THEN $f(x)$ HAS A RELATIVE MAXIMUM (MINIMUM) AT $x = \mathcal{E}$

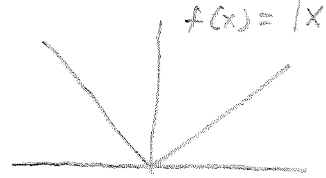
PROOF: ASSUME $f''(\mathcal{E}) < 0$

WITHIN SOME NEIGHBORHOOD OF $\mathcal{E}$ $\exists\ x < \mathcal{E} \ni f'(x) > f'(\mathcal{E}) = 0$ AND $x > \mathcal{E} \ni f'(x) < f'(\mathcal{E}) = 0$.

BY FIRST DIRIVITIVE TEST, $f(x)$ HAS REL. MAX AT $x = \mathcal{E}$

1) GIVE AN EXAMPLE OF A FUNCTION CONTINUOUS AT a, WHICH IS NOT DIFFERENTIABLE AT a.

$f(x) = |x|$



$\forall \varepsilon > 0 \ \exists \ \delta = \delta(\varepsilon) > 0 \ni |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

$|f(x) - f(a)| = ||x|| = |x| < \varepsilon \Rightarrow |x - a| < \varepsilon$

LET $\delta = \varepsilon$

$\Rightarrow f(x) = |x|$ IS CONT. AT $x = 0$

$\lim\limits_{\Delta x \to 0^+} \frac{\Delta y}{\Delta x} \neq \lim\limits_{\Delta x \to 0^-} \frac{\Delta y}{\Delta x} \Rightarrow$ A FUNCTION HAS A DERIVITIVE AT A POINT IFF IT HAS EQUAL DERIVITIVES FROM THE RIGHT & LEFT OF THE POINT

2) GIVE AN EXAMPLE OF A FUNCTION DIFFERENTIABLE AT $x = a$, BUT WHOSE DERIVITIVE IS DISCONTINUOUS AT $x = a$

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad a = 0$$

$$f'(0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x^2 \sin\left(\frac{1}{\Delta x}\right)}{\Delta x} = \lim_{\Delta x \to 0} \Delta x \sin \frac{1}{\Delta x} = 0$$

$y = x^2 \sin \frac{1}{x} \Rightarrow \frac{dy}{dx} = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ WHEN $x \neq 0$

$$\lim_{x \to 0} \left[2x \sin \frac{1}{x} - \cos \frac{1}{x}\right] = \lim_{x \to 0} -\cos \frac{1}{x}$$

$\therefore \frac{dy}{dx}$ CAN APPROACH NO LIMIT

COURSE OBJECTIVES FOR MC_3

Material: Chap. IV of text. Sec 401-409 incl.

Performance required for MC_3.

1. To write in words and appropriate symbols, any definition given in Sec. 401-409.

2. To write a proof, complete with reasons for each step in the proof to each of the following.

   Thm I; p. 112
   Thm VIII, IX, X; p. 115 (for ~~limit~~ hint see prob 1, p. 125.)
   Thm I, p. 121
   Thm II, p. 128
   Thm p. 129.

3. (i) To be able to demonstrate that all integrable functions are bounded.

   (ii) To write in words all sufficient conditions for integrability as given in Chap. IV of text.

4. Problems: 1P 2, 5, 7, 9, 10, 16, 18.

5. To evaluate the following integrals, using the definition of the integral itself, (i.e. not by the Fundamental Thm), an appropriate net, and any necessary formulae for sums, such as

$$\sum_{k=1}^{n} k^{\alpha}, \quad \alpha \in PI, \quad \text{or} \quad \sum_{k=1}^{n} \sin(kx)$$

For example of Method see example, p. 115. I

Integrals to be evaluated.

$$\int_a^b x^n dx, \quad n = 1,2,3 \quad , \quad \int_0^{\pi/2} \sin x \, dx$$

(Note: It is not necessary that you memorize $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ etc. These will be provided as needed on a P.M.)

6. To prove that $f(x) = \begin{cases} 0, & x = 0 \\ \sin\left(\frac{1}{x}\right), & x \neq 0 \end{cases}$

   IS integrable on the interval $[0,1]$.

7. To given an example of a function with a countably infinite number of discontinuities an $[0,1]$, but which is still integrable; and to PROVE that the function you have given is integrable.

2?

8. Prob. 3, p. 135

9. Prob. 1,2,3,4,5,6,7,8 p. 129.

10. Prob. 15 - 21, p. 131.

(b) In the trapezoidal Rule for approximating a definite integral, why the factors $\frac{1}{2}$ on y_0 ē y_n, but 1 on $y_1, \dots y_{n-1}$.

Note: also that the MVT give an estimate on the magnitude of the error. Note in particular Error $\propto (\Delta x)^2$ so that decreasing (Δx) by a factor of $\frac{1}{2}$, decreases the Maximum error by ______?

(1)

P110 DEF: $\lim_{|\eta|\to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i = I \ni I \in (-\infty, +\infty)$ IFF $\forall\ \varepsilon > 0\ \exists$
$\delta > 0 \ni \forall\ |\eta| < \delta$ AND $\forall\ x_i \in [a_{i-1}, a_i],\ i = 1, 2, 3, \ldots, n;$
$\left| \sum_{i=1}^{n} f(x_i)\Delta x_i - I \right| < \varepsilon$

P111 DEF: $f(x)$ DEFINED ON $[a,b]$ IFF ABOVE LIMIT EXISTS.
CALLED "DEFINITE INTEGRAL" DENOTED
$$\int_a^b f(x)\,dx = \lim_{|\eta|\to 0} \sum_{i=0}^{n} f(x_i)\Delta x_i$$

P111 DEF: $b > a \Rightarrow \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
$$\int_a^a f(x)\,dx = 0$$

P.112 THEOREM I: IF $\lim_{|\eta|\to 0} \sum_{i=0}^{n} f(x_i)\Delta x_i$ EXISTS, IT IS UNIQUE

PROOF: ASSUME $\lim_{|\eta|\to 0} \sum_{i=0}^{n} f(x_i)\Delta x_i = I$; $\lim_{|\eta|\to 0} \sum_{i=0}^{n} f(x_i)\Delta x_i = J$
$\ni I > J$. LET $\varepsilon = \frac{1}{2}(I-J) \Rightarrow \exists\ \delta > 0 \ni |\eta| < \varepsilon$
$\Rightarrow \left|\sum_{i=0}^{n} f(x_i)\Delta x_i - I\right| < \varepsilon \Rightarrow I - \varepsilon < \sum_{i=0}^{n} f(x_i)\Delta x_i$
SIMILARLY $\left|\sum_{i=0}^{n} f(x_i)\Delta x_i - J\right| < \varepsilon \Rightarrow J + \varepsilon > \sum_{i=0}^{n} f(x_i)\Delta x_i$
$\Rightarrow I - \varepsilon < \sum_{i=0}^{n} f(x_i)\Delta x_i < J + \varepsilon \Rightarrow I - \varepsilon < J + \varepsilon$
$\Rightarrow \varepsilon > \frac{1}{2}(I-J)$, A COUNTERDICTION TO BASIC
ASSUMPTION Q.E.D.

P115 THEOREM VIII: A FUNCTION CONTINUOUS OVER A CLOSED INTERVAL IS INTEGRABLE

PROOF: $f(x)$ CONT. ON $[a,b] \Rightarrow f(x)$ UNIFORMLY CONT. ON $[a,b]$
$\Rightarrow \forall\ \varepsilon' > 0,\ \exists\ \delta > 0 \ni |x' - x''| < \delta \Rightarrow |f(x') - f(x'')| < \varepsilon'$
LET $\varepsilon' = \frac{\varepsilon}{b-a} > 0 \Rightarrow |f(x') - f(x'')| < \frac{\varepsilon}{b-a}$. LET $|\eta| < \delta$
LET $\tau_i = \text{MAX}\,[a_{i-1}, a_i] = f(x_i')$; $\sigma_i = \text{MIN}\,[a_{i-1}, a_i] = f(x_i'')$; $i = 1, 2, \ldots n$
LET $\tau(x) = \tau_i \ni \sigma(x) = \sigma_i$ FOR $x \in (a_{i-1}, a_i)$; $i = 1, 2, \ldots, n$
$\sigma(x) = \tau(x) = f(a_i)$ FOR $i = 0, 1, 2, \ldots, n$, THEN $\sigma(x) \le f(x) \le \tau(x)$
$$\Rightarrow \int_a^b [\tau(x) - \sigma(x)]\,dx = \sum_{i=1}^{n} (\tau_i - \sigma_i)\Delta x_i$$
$$= \sum_{i=1}^{n} |f(x_i') - f(x_i'')|\,\Delta x_i < \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \Delta x_i = \varepsilon$$
$\forall\ \varepsilon > 0\ \exists$ STEP FUNCTIONS $\sigma(x) \ni \tau(x) \ni \sigma(x) \le f(x) \le \tau(x)$ FOR $a \le x \le b$
AND $\int_a^b [\tau(x) - \sigma(x)]\,dx < \varepsilon \Rightarrow f(x)$ IS INTERABLE ON $[a,b]$

IX

P115 THEOREM: $f(x)$ BOUNDED ON $[a,b]$, AND CONTINUOUS THE, SAVE AT A FINITE NUMBER OF POINTS, IS INTEGRABLE THERE

ie) $f(x)$ DISCONTINUOUS AT $(b_0, b_1, b_2, \ldots, b_n) \ni n \in P.I.$

$$\Rightarrow \int_{b_0}^{b_n} f(x)\,dx = \sum_{i=1}^{n} \int_{b_{i-1}}^{b_i} f(x)\,dx$$

PROOF: BY MATHEMATICAL INDUCTION

1) ESTABLISH FOR $f(x)$ DEFINED AND BOUNDED ON $[a,b]$, AND CONTINUOUS ON THE INTERIOR.

$\therefore \forall\, c,d \ni a < c < d < b,\ f(x)$ IS CONT. ON $[c,d]$

$\Rightarrow f(x)$ IS INTEGRABLE ON $[a,b]$

2) $\int_{b_0}^{b_n} f(x)\,dx \overset{?}{=} \sum_{i=1}^{n} \int_{b_{i-1}}^{b_i} f(x)\,dx$ WITH b_i A POINT OF DISCONTINUITY. W.L.O.G., LET $b_i > b_{i-1}$

$$\int_{b_0}^{b_{n+1}} f(x)\,dx = \sum_{i=1}^{n} \int_{b_{i-1}}^{b_i} f(x)\,dx + \int_{b_n}^{b_{n+1}} f(x)\,dx$$

$$b_i > b_{i-1} \Rightarrow \int_{b_0}^{b_{n-1}} f(x)\,dx = \sum_{i=1}^{n+1} f(x)\,dx \qquad \text{QED}$$

P120 DEFINITION: STEP FUNCTION: A FUNCTION DEFINED ON $[a,b]$ THAT IS CONSTANT IN THE INTERIOR OF EACH SUBINTERVAL OF SOME NET ON $[a,b]$

P121 THEOREM I: ANY STEP FUNCTION IS INTEGRABLE, ie

$$\int_a^b \sigma(x)\,dx = \sum_{i=1}^{m} \sigma_i (\alpha_i - \alpha_{i-1})$$

PROOF: BY MATHEMATICAL INDUCTION

1) LET $\sigma(x) = k$ ON (a,b). CHANGING FINITE # OF PTS. REDEFINE END POINTS $\ni \sigma(x) = k$ ON $[a,b]$

$$\Rightarrow \int_a^b \sigma(x)\,dx = k(b-a)$$

2) $\int_{x_0}^{x_n} \sigma(x)\,dx = \sum_{i=1}^{m} \sigma_i (\alpha_i + \alpha_{i-1})$

$$\int_{x_0}^{x_m} \sigma(x)\,dx + \int_{x_m}^{x_{m+1}} \sigma(x)\,dx = \sum_{i=1}^{m} \sigma_i (\alpha_i + \alpha_{i-1}) + \sigma_{m+1}(\alpha_{m+1} - \alpha_m)$$

SINCE $m_{n+1} > m_n$

$$\int_{x_0}^{x_{m+1}} \sigma(x)\,dx = \sum_{i=0}^{m+1} \sigma_i (\alpha_i + \alpha_{i-1})$$

II

P 128 FUNDEMENTAL THEOREM OF INTEGRAL CALCULUS: $f(x)$ CONT. ON $[a,b]$, AND $F(x)$ IS ANY PRIMITIVE OF $f(x)$ ON $[a,b]$

$$\Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$$

PROOF: $\frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}F(x)$

$\Rightarrow \int_a^x f(t)\,dt - F(x) \equiv C$

$\int_a^a f(t)\,dt - F(a) \equiv C \Rightarrow 0 - F(a) = C \Rightarrow C = {}^-F(a)$

$\Rightarrow \int_a^x f(t)\,dt - F(x) = {}^-F(a) \Rightarrow \int_a^x f(t)\,dt = F(x) - F(a)$

$b = x \Rightarrow \int_a^b f(t)\,dt = F(b) - F(a)$ QED

FIRST MEAN VALUE THEOREM: IF $f(x)$ CONT. ON $[a,b] \Rightarrow \exists\ \varepsilon \in (a,b) \ni \int_a^b f(x)\,dx = f(\varepsilon)\cdot(b-a)$

PROOF: LET $m = \text{MIN } f(x)$ & $M = \text{MAX } f(x)$

($f(x)$ CONT. ON $[a,b] \Rightarrow f(x)$ ASSUMES MAX & MIN ON $[a,b]$

$\therefore\ m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

$\Rightarrow m \leq \frac{\int_a^b f(x)\,dx}{b-a} \leq M$

$f(x)$ CONT. ON $[a,b] \Rightarrow f(x)$ ASSUMES ALL VALUES BETWEEN m & M

$\therefore \exists\ \varepsilon \in [a,b] \ni m \leq f(\varepsilon) \leq M \ni$

$f(\varepsilon) = \left[\int_a^b f(x)\,dx\right] / (b-a)$

$\Rightarrow \int_a^b f(x)\,dx = f(\varepsilon)\cdot(b-a)$

P 118 * EX 7 THEOREM: IF $f(x) = f(-x)$, & $f(x)$ INTEGREBLE ON $[0,a]$,

THEN $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ ASSUME $^-a < 0 < a$

PROOF: $\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx$

$= -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx$

LET $x = {}^-u \Rightarrow \int_{-a}^a f(x)\,dx = \int_0^a f(-u)\,du + \int_0^a f(x)\,dx$

$\Rightarrow \int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$

EX 7b

P118 * THEOREM: IF $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ AND $f(x) = f(-x)$ ON $(-\infty,\infty)$, THEN $\int_{-b}^{-a} f(x)\,dx = \int_{a}^{b} f(x)\,dx$

PROOF: ASSUME $-b < -a < 0 < a < b$

$$\int_{-b}^{b} f(x)\,dx = 2\int_{0}^{b} f(x)\,dx$$

$$\Leftrightarrow \int_{-b}^{-a} f(x)dx + \int_{-a}^{a} f(x)dx + \int_{a}^{b} f(x)dx = 2\int_{0}^{a} f(x)dx + 2\int_{a}^{b} f(x)dx$$

$$\Leftrightarrow \int_{-b}^{-a} f(x)dx + 2\int_{0}^{a} f(x)dx + \int_{a}^{b} f(x)dx = 2\int_{0}^{a} f(x)dx + 2\int_{a}^{b} f(x)dx$$

$$\Leftrightarrow \int_{-b}^{-a} f(x)\,dx = \int_{a}^{b} f(x)\,dx$$

Q.E.D.

EX 9

* THEOREM: $f(x) = f(-x)$ AND $f(x) = -f(-x) \Rightarrow f(x) = 0$

P118 PROOF: ASSUME $f(x) \neq 0$

$$2f(x) = f(-x) - f(-x) = 0$$

Q.E.D.

Ex 10

* THEOREM: ANY FUNCTION WHOSE DOMAIN CONTAINS

P118 THE NEGATIVE OF EVERY ONE OF ITS MEMBER IS UNIQUELY REPRESENTABLE AS THE SUM OF AN EVEN AND ODD FUNCTION

PROOF: $f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$

LET $f_e(x) = \frac{1}{2}[f(x) + f(-x)]$

$$f_e(-x) = \frac{1}{2}[f(-x) + f(x)] = \frac{1}{2}[f(x) + f(-x)]$$

$\Rightarrow f_e(x)$ IS AN EVEN FUNCTION

LET $f_o(x) = \frac{1}{2}[f(x) - f(-x)]$

$$-f_o(-x) = \frac{1}{2}[-f(-x) + f(x)] = \frac{1}{2}[f(x) - f(-x)]$$

$\Rightarrow f_o(x)$ IS AN ODD FUNCTION

IS THIS REPRESENTATION UNIQUE?

LET $f(x) = f_{e_1}(x) + f_{o_1}(x)$ AND $f(x) = f_{e_0}(x) + f_{o_0}(x)$

$\Rightarrow f_{e_1}(x) \neq f_{e_0}(x)$ OR $f_{o_1}(x) \neq f_{o_0}(x)$

$$f_{e_1}(x) + f_{o_1}(x) = f_{e_0}(x) + f_{o_0}(x)$$

$$f_{e_1}(x) - f_{o_1}(-x) = f_{e_0}(x) - f_{o_0}(x)$$

ADDING $\Rightarrow 2f_{e_1}(x) = 2f_{e_0}(x) \Rightarrow f_{e_1}(x) = f_{e_0}(x)$

$\Rightarrow f_{o_1}(x) = f_{o_0}(x)$

Q.E.D.

P128 DEF: A PRIMITIVE, ANTIDERIVATIVE OR INDEFINITE INTEGRAL OF A GIVEN FUNCTION IS ANY FUNCTION WHOSE DERIVATIVE IS THE GIVEN FUCTION.

P129 THEOREM: IF $f(v)$ IS A CONTINUOUS FUNCTION OF v, AND $v(x)$ IS A CONT. FUNCTION OF x, THEN

$$\int f[v(x)]\,v'(x)\,dx = \int f[v(x)]\,d[v(x)] = \int f(v)\,dv$$

IF $v(a) = c$ & $v(b) = d \Rightarrow \int_a^b f[v(x)]\,v'(x)\,dx = \int_c^d f(v)\,dv$

PROOF: $v'(x) \equiv \frac{d}{dx} v(t) \Rightarrow v'(x)\,dx = d[v(t)]$

$$\therefore \int f[v(x)]\,v'(x)\,dx = \int f[v(x)]\,d[v(x)]$$

$$\int_a^b f(v(x))\,d[v(x)] = F[v(b)] - F[v(a)] =$$

$$\int_c^d f(v)\,dv \qquad = F(d) - F(c)$$

P129 * EX1 THEOREM: IF a & x ARE ELEMENTS OF CONT. INERVAL I

$$\Rightarrow \frac{d}{dx}\int_a^x f(t)\,dt = f(x)\ ;\ \frac{d}{dx}\int_x^a f(t)\,dt = -f(x)$$

PROOF: $\frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}\left[F(x) - F(a)\right]$

$$= \frac{d}{dx} F(x) = f(x)$$

$$\frac{d}{dx}\int_x^a f(t)\,dt = -\frac{d}{dx}\int_a^x f(t)\,dt$$

$$= -f(x)$$

P129 * EX2 THEOREM: IF $u(x)$ & $v(x)$ ARE DIFF. FUNCTIONS ELEMENTS IN AN INTERVAL I ON WHICH $f(t)$ IS CONT, THEN $\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f[v(x)]\,v'(x) - f[u(x)]\,u'(x)$

PROOF: $\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = \frac{d}{dx}\left[F[v(x)] - F[u(x)]\right]$

$$= v'(x)\,f[v(x)] - u'(x)\,f[u(x)]$$

EXERCISES: 3-8 P129

1) $\frac{d}{dx}\int_a^b \sin x^2\,dx = 0$

2) $\frac{d}{dx}\int_x^b \sin t^2\,dt = -\sin x^2$

3) $\frac{d}{dx}\int_a^x \sin t^2\,dt = \sin x^2$

4) $\frac{d}{dx}\int_0^{x^3} \sin t^2\,dt = 3x^2 \sin x^6$

5) $\frac{d}{dx}\int_{x^3}^{x^4} \sin t^2\,dt = 4x^3 \sin x^8 - 3x^2 \sin x^6$

6) $\frac{d}{dx}\int_{\sin^2 x}^{\sin x^2} \sin t^2\,dt = 2x \cos x^2 \sin(\sin x^2)^2 - 2 \sin x \cos x \sin(\sin^4 x)$

ESTABLISH REDUCTION FORMULA FOR: $\int \tan^n x\,dx$

ANSWER: $\frac{d}{dx}\tan^p x = p\sec^2 x \tan^{p-1} x$

$$= p(1+\tan^2 x)\tan^{p-1} x$$

$$= p\tan^{p-1} x + p\tan^{p+1} x$$

$$\frac{\tan^p x}{p} = \int \tan^{p-1} x\,dx + \int \tan^{p+1} x\,dx$$

Let $n = p+1 \Rightarrow p-1 = n-2 \Rightarrow p = n-1$

$$\frac{\tan^{n-1} x}{n-1} = \int \tan^{n-2} x\,dx + \int \tan^n x\,dx$$

$$\Rightarrow \int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx \quad \text{FOR } n \neq 1$$

SHOW THAT $f(x)$ DEFINED ON $[0,1]$ AS

$$f(x)\begin{cases} = 1 & \forall \text{ RATIONAL } x \\ = 0 & \forall \text{ IRRATIONAL } x \end{cases}$$

HAS NEITHER INTEGRAL NOR PRIMITIVE THERE

PROOF: 1) LET $F(x)$ BE A PRIMITIVE OF $f(x) \Rightarrow F'(x) = f(x)$

$\therefore f(x)$ ASSUMES EVERY # BETWEEN ANY 2 OF ITS VALU[E]

$f(0)=1$; $f(\frac{1}{\sqrt{2}})=0$ BUT $\exists$ NO $f(x) \ni f(x) \in (0,1)$

2) LET $\sigma(x) = K$ FOR $[0,1] \ni K \leq 0$

$\tau(x) = L$ FOR $[0,1] \ni L \geq 1 \Rightarrow \sigma(x) \leq f(x) \leq \tau(x)$

LET $\varepsilon = 1 \Rightarrow \int_0^1 [\tau(x) - \sigma(x)]\,dx \geq \varepsilon$

$\therefore f(x)$ IS NOT INTEGRABLE

SHOW THAT THOUGH $\int_a^x f(t)\,dt$ IS ALWAYS A PRIMITIVE OF $f(x)$, NOT EVERY PRIMITIVE OF $f(x)$ CAN BE EXPRESSED AS $\int_a^x f(t)\,dt$

A) $F(x) = \int_a^x f(t)\,dt \Rightarrow F'(t) = F'(x) - F'(a) = f(x)$

B) LET $f(x) = \cos x \ni F(x) = \sin x + K$

$$F(x) = \sin x + K = \int_a^x f(t)\,dt = F(x) - F(a)$$

$$\Rightarrow F(a) = 0. \text{ HOWEVER } F(a) = K$$

QED

SHOW THAT $f(x)=\begin{cases}0 & \forall\, x\in[0,1]\\ 1 & \forall\, x\in(1,2]\end{cases}$ IS INTEGRABLE ON $[0,2]$, BUT HAS NO PRIMITIVE THERE

1) $\int_0^2 f(x)dx = \int_0^1 0\,dx + \int_1^2 1\,dx = 1$

2) ASSUME $F(X)$ IS PRIMITIVE OF $f(x) \Rightarrow F'(x)=f(x)$

$\Rightarrow f(x)$ ASSUMES ALL THE NUMBERS BETWEEN ANY TWO OF ITS VALUES. $f(0)=0$; $f(2)=1$; BUT $\nexists$ NO $f(x)\in(0,1)$

SHOW THAT $F(x)=x^2 \sin\frac{1}{x^2}$ $[F(0)=0]$ HAS UNBOUNDED DERIVATIVE $f(x)\in[0,1] \Rightarrow$ A FUNCTION MAY HAVE A PRIMITIVE WITHOUT BEING INTEGRABLE:

$f(x)=F'(x)=2x\sin\frac{1}{x^2}-\frac{1}{x}\cos\frac{1}{x^2}$

$f(0^+)=-\infty \Rightarrow f(x)$ IS NOT INTEGRABLE, BUT HAS A PRIMITIVE

PROVE $f(x)=\begin{cases}0 & x=0\\ \sin\frac{1}{x} & x\neq 0\end{cases}$ IS INTEGRABLE ON $[0,1]$

PROOF: $f(x_0)$ EXISTS $\forall\, x\in[0,1]$

$\lim_{x\to x_0} f(x)=f(x_0) \quad \forall\, x\in(0,1]$

$\therefore \sin\frac{1}{x}$ IS CONT. ON $(0,1]$

$\Rightarrow \forall [c,d] \ni 0<c<d<1$ IS CONT. $\Rightarrow f(x)$ IS INTEGRABLE ON $[c,d]$

$-1\le f(x)\le 1 \Rightarrow f(x)$ IS BOUNDED

$\Rightarrow f(x)$ IS INTEGRABLE ON $[0,1]$

THEOREM: A FUNCTION DEFINED AND MONOTONIC ON A CLOSED INTERVAL IS INTEGRABLE THERE

PROOF: FOR DEFINITNESS, LET $f(x)$ BE MONOTONICALLY INCREASING ON $[a, b]$. FOR A GIVEN η

LET $\sigma(x) = f(a_{i-1})$ FOR $a_{i-1} \leq x < a_i$

$[\sigma(b) = f(b)]$ AND $\tau(x) = f(a_i)$ FOR $a_{i-1} < x \leq a_i$

$[\tau(a) = f(a)]$; $i = 1, 2, 3, \ldots, n$

THEN $\sigma(x) \leq f(x) \leq \tau(x)$

$$\int_a^b [\tau(x) - \sigma(x)]dx = \sum_{i=1}^{n} [f(a_i) - f(a_{i-1})] \Delta x_i$$

$$\leq |\eta| \sum_{i=1}^{n} [f(a_i) - f(a_{i-1})] = |\eta| \cdot [f(b) - f(a)]$$

LET $\delta(\varepsilon) = \varepsilon / [f(b) - f(a)] \Rightarrow |\eta| < \dfrac{\varepsilon}{[f(b) - f(a)]}$

$$\Rightarrow |\eta| \cdot [f(b) - f(a)] < \varepsilon$$

$$\Rightarrow \int_a^b [\tau(x) - \sigma(x)]dx < \varepsilon \qquad \text{Q.E.D.}$$

PROVE FOR $m = 2, 3,$ OR 4

$$\int_a^b x^m dx = \frac{1}{m+1} \left[b^{m+1} - a^{m+1}\right]$$

LET $a_i = a + \dfrac{i(b-a)}{n}$; $x_i = a_i$; $\Delta x_i = \dfrac{b-a}{n}$

$$\sum_{i=0}^{n} \left[a + \frac{i(b-a)}{n}\right]^2 \frac{(b-a)}{n} = \frac{b-a}{n} \sum_{i=0}^{n} \left[a + \frac{i(b-a)}{n}\right]^2$$

$$\frac{b-a}{n} \frac{1}{2} \left[a + \frac{(b-a)}{n}\right]\left[1 + a + \frac{(b-a)}{n}\right]$$

ARG!

I) DEFINITIONS

1) a) NET: ON A FIXED CLOSED INTERVAL, A FINITE SET OF PTS; $[a,b]: a=a_0<a_1<a_2<\ldots<a_n=b$

b) SUBINTERVAL: $[a_{i-1},a]$, $i=1,2,3,\ldots,n$ ON $[a,b]$ FOR NET η

c) NORM: $|\eta|$: MAXIMUM LENGTH OF SUBINTERVAL

2) $\lim_{|\eta|\to 0}\sum_{i=1}^{n} f(x_i)\Delta x_i = I \ni I\in(-\infty,+\infty)$ IFF $\forall\, \varepsilon>0,\ \exists\, \delta=\delta(\varepsilon)>0 \ni \forall |\eta|<\delta$ AND $\forall\, x_i\in[a_{i-1}\le x_i\le a_i]$, $i=1,2,3,\ldots,n \Rightarrow \left|\sum_{i=1}^{n} f(x_i)\Delta x_i - I\right|<\varepsilon$

3) $f(x)$ DEFINED ON $[a,b]$ IS INTEGRABLE THERE IFF $\lim_{|\eta|\to 0}\sum_{i=1}^{n} f(x_i)\Delta x_i$ EXISTS AND IS FINITE. IS CALLED THE DEFINITE INTEGRABLE: $\int_a^b f(x)dx = \lim_{|\eta|\to 0}\sum_{i=1}^{n} f(x_i)\Delta x_i$

4) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ IF $b<a$; $\int_a^a f(x)dx=0$

5) STEP FUNCTION: A FUNCTION, DEFINED ON $[a,b]$, THAT IS CONSTANT IN THE INTERIOR OF EACH SUBINTERVAL OF SOME NET ON $[a,b]$

6) PRIMITIVE: ANTIDERIVATIVE: INDEFINITE INTEGRAL: OF A FUNCTION IS ANY FUNCTION WHOSE DERIVITIVE IS THE GIVEN FUNCTION

II) THEOREMS

1) IF $\lim_{|\eta|\to 0}\sum_{i=0}^{n} f(x_i)\Delta x_i$ EXISTS, IT IS UNIQUE

PROOF: $\lim_{|\eta|\to 0}\sum_{i=0}^{n} f(x_i)\Delta x_i$ EXISTS $\Rightarrow \forall\, \varepsilon>0\ \exists\, \delta>0 \ni \forall |\eta|<\delta$ AND $\forall\, x_i\in[a_{i-1},a_i]$, $i=1,2,3,\ldots,n \Rightarrow \left|\sum_{i=0}^{n} f(x_i)\Delta x_i - I\right|<\varepsilon$

ASSUME $\lim_{|\eta|\to 0}\sum_{i=1}^{n} f(x_i)\Delta x_i = I$ AND $\lim_{|\eta|\to 0}\sum_{i=1}^{n} f(x_i)\Delta x_i = J$

W.L.O.G., LET $I>J$. LET $\varepsilon=\frac{1}{2}(J-I)>0$

$\Rightarrow \exists\, \delta>0 \ni \left|\sum_{i=0}^{n} f(x_i)\Delta x_i - I\right|<\varepsilon$

$\Rightarrow -\varepsilon<\sum_{i=0}^{n} f(x_i)\Delta x_i - I<\varepsilon \Rightarrow I-\varepsilon<\sum_{i=0}^{n} f(x_i)\Delta x_i<I+\varepsilon$

SIMILARLY : $J-\varepsilon<\sum_{i=0}^{n} f(x_i)\Delta x_i<J+\varepsilon$

$\Rightarrow I-\varepsilon<\sum_{i=0}^{n} f(x_i)\Delta x_i<J+\varepsilon \Rightarrow I-\varepsilon<J+\varepsilon$

$\Rightarrow \varepsilon>\frac{1}{2}(J-I)$, A COUNTERDICTION TO BASIC ASSUMPTION Q.E.D.

2) A FUNCTION CONTINUOS ON A CLOSED INTERVAL IS INTEGRABLE THERE

PROOF: $f(x)$ DEFINED ON $[a,b]$ IS INTEGRABLE THERE IFF $\forall\, \varepsilon>0$ $\exists$ STEP FUNCTIONS $\sigma(x)$ & $\tau(x) \ni \sigma(x)\le f(x)\le \tau(x)\ \forall\, x\in[a,b]$, AND $\int_a^b[\tau(x)-\sigma(x)]dx<\varepsilon$

$f(x)$ CONT. ON $[a,b] \Rightarrow f(x)$ IS UNIFORMY CONTINUOUS ON $[a,b]$

$\Rightarrow \forall\, \varepsilon'>0, \exists\, \delta>0 \ni \forall\, x',x''\in[a,b]$, $|x'-x''|<\delta \Rightarrow |f(x')-f(x'')|<\varepsilon'$

LET $\varepsilon'=\frac{\varepsilon}{b-a}>0 \Rightarrow |f(x')-f(x'')|<\frac{\varepsilon}{b-a}$. LET $|\eta|<\delta$

DEFINE $\sigma_i=\text{MAX}[a_{i-1},a_i]=f(x_i')$; $\tau_i=\text{MIN}[a_{i-1},a_i]=f(x_i'')$; $i=1,2,3\ldots n$

DEFINE $\tau(x)=\tau_i$ AND $\sigma(x)=\sigma_i\ \forall\, x\in(a_{i-1},a_i)$; $i=1,2,3,\ldots,n$

$\sigma(x)=\tau(x)=f(a_i)$ $\quad i=0,1,2,\ldots n \Rightarrow \sigma(x)\le f(x)\le \tau(x)$

$\forall\, x\in[a,b]$.

$\int_a^b[\tau(x)-\sigma(x)]dx=\sum_{i=1}^{n}(\tau_i-\sigma_i)\Delta x_i=\sum_{i=1}^{n}|f(x_i')-f(x_i'')|\Delta x$

$\le \frac{\varepsilon}{b-a}\sum_{i=0}^{n}\Delta x_i=\varepsilon$ Q.E.D.

3) A FUNCTION DEFINED AND BOUNDED ON A CLOSED INTERVAL AND CONTINUOUS THERE, EXCEPT FOR A FINITE NUMBER OF POINTS, IS INTEGRABLE THERE. ie IF $f(x)$ IS CONTINUOUS ON $[b_0, b_n]$ EXCEPT AT PTS. b_i; $i = 1,2,3,\ldots,n-1 \ni b_{i-1} < b_i$, (MAY BE DISC. AT b_0 OR b_n)

THEN $\int_a^b f(x)dx = \sum_{i=1}^{n} \int_{b_{i-1}}^{b_i} f(x)dx$

PROOF: BY MATHEMATICAL INDUCTION AND ESTABLISH FOR $f(x)$ DEFINED AND BOUNDED ON $[a,b]$ AND CONTINUOUS ON $(a,b) \Rightarrow \therefore \forall c,d \in (a,b)$ ie $a<c<d<b$, $f(x)$ IS CONT ON $[c,d] \Rightarrow f(x)$ IS INTEGRABLE ON $[a,b]$

$$\int_{b_0}^{b_n} f(x)dx = \sum_{i=1}^{n} \int_{b_{i-1}}^{b_i} f(x)dx$$

$$\Rightarrow \int_{b_0}^{b_{n+1}} f(x)dx = \sum_{i=1}^{n+1} \int_{b_{i-1}}^{b_i} f(x)dx \overset{?}{=} \sum_{i=1}^{n} \int_{b_{i-1}}^{b_i} f(x)dx + \int_{b_n}^{b_{n+1}} f(x)dx$$

$$= \sum_{i=1}^{n+1} \int_{b_{i-1}}^{b_i} f(x)dx$$

Q.E.D.

4) A FUNCTION DEFINED AND MONOTONIC ON $[a,b]$ IS INTEGRABLE (INCR)

PROOF: $f(x)$ DEFINED ON $[a,b]$ IS INTEGRABLE THERE IFF $\forall \varepsilon > 0$ $\exists$ STEP FUNCTIONS $\sigma(x)$ & $\tau(x) \ni \sigma(x) \le f(x) \le \tau(x)$ $\forall$ $x \in [a,b]$ AND $\int_a^b [\tau(x) - \sigma(x)]dx < \varepsilon$. LET $|\eta| < \frac{\varepsilon}{f(b)-f(a)}$ FOR A GIVEN NET

DEFINE STEP FUNCTIONS $\sigma(x) = f(a_{i-1})$ $\forall$ $x \in [a_{i-1}, a_i)$; $(\sigma(b) = f(b))$

$\tau(x) = f(a_i)$ $\forall$ $x \in (a_{i-1}, a_i]$; $(\tau(a) = f(a))$ $i = 1,2,3,\ldots,n$

$\Rightarrow \sigma(x) \le f(x) \le \tau(x)$ AND $\int_a^b [\tau(x) - \sigma(x)]dx = \sum_{i=1}^{n} [f(a_i) - f(a_{i-1})]\Delta x_i$

$\le |\eta| \cdot \sum_{i=1}^{n} [f(a_i) - f(a_{i-1})] = |\eta| \cdot [f(b) - f(a)] < \varepsilon$ Q.E.D.

5) ANY STEP FUNCTION IS INTEGRABLE

ie $\int_a^b \sigma(x) = \sum_{i=1}^{m} \sigma_i (\alpha_i - \alpha_{i-1})$

PROOF: BY MATHEMATICAL INDUCTION

$\sigma(x) = k$ $\forall x \in (a,b)$ MAY REDEFINE 2 ENDPOINTS WITHOUT CHANGING INTEGRABILITY. LET $\sigma(a) = \sigma(b) = k$

$$\Rightarrow \sigma(x) = k \ \forall x \in [a,b] \Rightarrow \int_a^b \sigma(x)dx = k(b-a)$$

$$\int_a^b \sigma(x) = \int_{\alpha_0}^{\alpha_m} \sigma(x)dx = \sum_{i=1}^{m} \sigma_i(\alpha_i - \alpha_{i-1})$$

$$\alpha_{m+1} > \alpha_m \Rightarrow \int_{\alpha_0}^{\alpha_m} \sigma(x)dx + \int_{\alpha_m}^{\alpha_{m+1}} \sigma(x)dx = \int_{\alpha_0}^{\alpha_{m+1}} \sigma(x) = \sum_{i=1}^{m+1} \sigma_i(\alpha_i - \alpha_{i-1})$$

Q.E.D.

6) FUNDEMENTAL THEOREM OF INTEGRAL CALCULUS:

$f(x)$ IS CONT. ON $[a,b]$, AND IF $F(x)$ IS ANY PRIMITIVE OF $f(x)$, THEN $\int_a^b f(x)dx = F(b) - F(a)$

PROOF: $\frac{d}{dx}\int_a^x f(t)dt = F'(x)$

$$\Rightarrow \int_a^x f(t)dt - F(x) = C$$

$$x = a \Rightarrow \int_a^a f(t)dt - F(a) = C \Rightarrow -F(a) = C$$

$$\Rightarrow \int_a^x f(t)dt = F(x) - F(a)$$

LET $b \equiv x$

$$\Rightarrow \int_a^b f(t)dt = F(b) - F(a)$$

Q.E.D.

7) IF $f(v)$ IS A CONTINUOS FUNCTION OF V & $V(x)$ IS A CONTINUOUS FUNCTION OF X, THEN

$$\int f[V(x)]V'(x)\,dx = \int f[V(x)]\,d[V(x)] = \int f(v)\,dV$$

~~AND~~ IF $V(a)=c$ & $V(b)=d \Rightarrow \int_a^b f[V(x)]V'(x)\,dx = \int_c^d f(v)\,dV$

PROOF: $V'(x) = \frac{d}{dx}V(x) \Rightarrow V'(x)\,dx = d[V(x)]$

$$\Rightarrow \int [V(x)]\,V'(x)\,dx = \int f[V(x)]\,d[V(x)]$$

$$\int_a^b f[V(x)]\,d[V(x)] = F[V(b)] - F[V(a)]$$

$$= F(d) - F(c) = \int_c^d f(v)\,dV$$

III) 1) DEMONSTRATE ALL INTEGRABLE FUNCTIONS ARE ~~INTEGRABLE~~ BOUNDED

LET π BE AN ARBITRARY NET, AND LET $f(x)$ BE UNBOUNDED ON K^{TH} SUBINTERVAL $[a_{K-1}, a_K]$

$\Rightarrow \forall\ x_i \ni i \neq k$, THE POINT X_K CAN BE CHOSEN $\ni$

$$\lim_{|\pi| \to 0} \sum_{i=1}^{n} f(x_i)\,\Delta x_i$$

IS NUMERICALLY LARGER THAN ANY OTHER PREASSIGNED QUANTITY

2) SUFFICIENT CONDITIONS FOR INTEGRABILITY

a) $f(x)$ CONTINUOUS ON $[a,b]$

b) $f(x)$ BOUNDED ON $[a,b]$ AND CONTINUOUS THERE EXCEPT FOR A FINITE NUMBER OF POINTS

c) $f(x)$ DEFINED AND MONOTONIC ON $[a,b]$

d) $f(x)$ A STEP FUNCTION ON $[a,b]$

e) IFF $\forall \varepsilon > 0 \exists$ STEP FUNCTIONS $\tau(x)$ & $\sigma(x) \ni \sigma(x) \leq f(x) \leq \tau(x)$ FOR $a \leq x \leq b$ AND $\int_a^b [\tau(x) - \sigma(x)]\,dx < \varepsilon \Rightarrow f(x)$ INTEGR.

IV) 1) ASSUMING $f(x)$ IS INTEGRABLE ON $[a.b]$. PROVE

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx \leq K(b-a), \text{ WHEN } |f(x)| \leq K$$

PROOF:

$$-|f(x)| \leq f(x) \leq |f(x)|$$

$$\int_a^b -|f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$$

$$\Rightarrow \left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

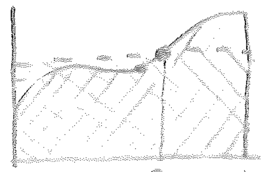
$$|f(x)| \leq K$$

$$\Rightarrow \int_a^b |f(x)|\,dx \leq \int_a^b K\,dx = K(b-a)$$

$$\therefore \left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx \leq K(b-a)$$

2) FIRST MEAN VALUE THEOREM: IF $f(x)$ IS CONT. ON $[a,b]$, $\exists\ \mathcal{E} \in (a,b) \ni \int_a^b f(x)dx = f(\mathcal{E})\cdot(b-a)$

PROOF:



LET $M = \text{MAX}(a,b)$ & $m = \text{MIN}(a,b)$

$\Rightarrow m \leq f(x) \leq M \quad \forall\ x \in [a,b]$

$\Rightarrow \int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx$

$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$. ASSUME $b>a$

$\Rightarrow m \leq \left[\left[\int_a^b f(x)dx\right]/(b-a)\right] \leq M$

ASSUME $m \neq M$ (PROOF WOULD THUS BE TRIVIAL)

$\Rightarrow m < \dfrac{\int_a^b f(x)dx}{b-a} < M$. A FUNCTION CONTINUOUS ON AN INTERVAL ASSUMES EVERY NUMBER BETWEEN ANY 2 VALUES IN THAT INTERVAL

$\Rightarrow \exists\ \mathcal{E} \in (m,M) \ni f(\mathcal{E}) = \dfrac{\int_a^b f(x)dx}{(b-a)}$

$\Rightarrow \int_a^b f(x)\,dx = f(\mathcal{E})\cdot(b-a)$

3) a) $f(x) = f(-x)\ \forall\ x \Rightarrow \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

PROOF: $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$

$= -\int_0^{-a} f(x)dx + \int_0^a f(x)dx$

$f(x) = f(-x) \Rightarrow f'(x) = -f'(-x)$

$\Rightarrow \int_{-a}^{a} f(x)dx = -\int_{-x=0}^{-x=a} f(-x)(-dx) + \int_0^a f(x)dx$

$= \int_0^a f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)dx$ QED

b) $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx \Rightarrow \int_{-b}^{-a} f(x)dx = \int_a^b f(x)dx$

PROOF: W.L.O.G. LET $-b < -a < 0 < a < b$

$\int_{-b}^{b} f(x)dx = \int_{-b}^{-a} f(x)dx + \int_{-a}^{a} f(x)dx + \int_a^b f(x)dx$

$\Leftrightarrow 2\int_0^b f(x)dx = \int_{-b}^{-a} f(x)dx + 2\int_0^a f(x)dx + \int_a^b f(x)dx$

$\Leftrightarrow 2\int_0^a f(x)dx + 2\int_a^b f(x)dx = \int_{-b}^{-a} f(x)dx + 2\int_0^a f(x)dx + \int_a^b f(x)dx$

$\Leftrightarrow \int_a^b f(x)dx = \int_{-b}^{-a} f(x)\,dx$

Q.E.D.

4) PROVE ONLY FUNCTION THAT IS BOTH EVEN & ODD IS 0

PROOF: $f(x)=f(-x)$; $f(x)=-f(-x)$

$\Rightarrow 2f(x)=f(-x)-f(-x)=0$

5) PROVE ANY FUNCTION WHOSE DOMAIN CONTAINS THE NEGATIVE OF EACH OF ITS MEMBERS IS UNIQUELY REPRESENTED BY AN ODD AND AN EVEN FUNCTION

PROOF: $f(x)=\frac{1}{2}[f(x)+f(-x)]+\frac{1}{2}[f(x)-f(-x)]$

LET $f_e(x)=\frac{1}{2}[f(x)+f(-x)]$

$f_e(-x)=\frac{1}{2}[f(-x)+f(x)]=f_e(x)\Rightarrow f_e(x)$ IS EVEN

LET $f_o(x)=\frac{1}{2}[f(x)-f(-x)]$

$-f_o(-x)=\frac{1}{2}[-f(-x)+f(x)]=f_o(x)\Rightarrow f_o(x)$ IS ODD

PROOF OF UNIQUENESS

LET $f(x)=f_{e_1}(x)+f_{o_1}(x)$ AND $f(x)=f_{e_0}(x)+f_{o_0}(x)$

$\ni f_{e_1}(x)\neq f_{o_1}(x)$ OR $f_{o_1}(x)\neq f_{o_0}(x)$

$f_{e_1}(x)+f_{o_1}(x)=f_{e_0}(x)+f_{o_0}(x)$

$f_{e_1}(x)-f_{o_1}(-x)=f_{e_0}(x)-f_{o_0}(-x)$

ADDING $\Rightarrow 2f_{e_1}(x)=2f_{e_0}(x)\Rightarrow f_{e_1}(x)=f_{e_0}(x)\Rightarrow f_{o_1}(x)=f_{e_1}(x)$ Q.E.D.

6) EVALUATE $\lim_{n\to+\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}\right)$

$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}\right)=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\dots+\frac{1}{1+\frac{n}{n}}\right)$

$=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+\frac{i}{n}}$

DEFINE $\Delta x_i=\frac{1}{n}\Rightarrow |n|\to 0$ AS $n\to\infty$; $\frac{i}{n}=i\,\Delta x_i=x\Rightarrow f(x)=\frac{1}{1+x}$

$\to\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+\frac{i}{n}}=\lim_{|n|\to 0}\sum_{i=0}^{n}\left(\frac{1}{1+x_i}\right)\Delta x_i=\int_0^1\frac{1}{x+1}\,dx$

7) PROVE $\forall\, a,x\in I$ (INTERVAL) OVER WHICH $f(x)$ IS CONTINUOUS

$\Rightarrow \frac{d}{dx}\int_a^x f(t)\,dt=f(x)$; $\frac{d}{dx}\int_x^a f(t)\,dt=-f(x)$

PROVE: $\frac{d}{dx}\int_a^x f(t)\,dt=\frac{d}{dx}[F(x)-F(a)]=F'(x)-F'(a)=f(x)$

$\frac{d}{dx}\int_x^a f(x)\,dt=\frac{d}{dx}[F(a)-F(x)]=F'(a)-F'(x)=-f(x)$

8) PROVE THAT IF $u(x)$ AND $v(x)$ ARE DIFFERENTIABLE FUNCTIONS WHOSE VALUES LIE IN AN INTERVAL I, WHICH $f(t)$ IS CONTINUOUS THROUGHOUT THEN

$\frac{d}{dx}\int_{u(x)}^{v(x)}f(t)\,dt=f[v(x)]v'(x)-f[u(x)]u'(x)$

PROOF: $\frac{d}{dx}\int_{u(v)}^{v(x)}f(t)\,dt=\frac{d}{dx}\left[F[v(x)]-F[u(x)]\right]=f[v(x)]v'(x)-f[u(x)]u'(x)$

9) EXCERCISES:

a) $\frac{d}{dx}\int_a^b \sin x^2\,dx=0$

b) $\frac{d}{dx}\int_a^x \sin t^2\,dt=\sin x^2$

c) $\frac{d}{dx}\int_x^b \sin t^2\,dt=-\sin x^2$

d) $\frac{d}{dx}\int_a^{x^3}\sin t^2\,dt=3x^2\sin x^6$

e) $\frac{d}{dx}\int_{x^3}^{x^4}\sin t^2\,dt=4x^3\sin x^8-3x^2\sin x^6$

f) $\frac{d}{dx}\int_{\sin^2 x}^{\sin x^2}\sin t^2\,dt=2x\cos x^2\sin[\sin^2 x^2]-2\sin x\cos x\sin[\sin^4 x]$

10) SHOW $f(x)$ DEFINED ON $[0,1] \ni f(x) = \begin{cases} 1 & \forall \text{ RAT } x \\ 0 & \forall \text{ IRR } x \end{cases}$ HAS NEITHER INTEGRAL OR PRIMITIVE THERE

PROOF: LET $F(x)$ BE A PRIMITIVE OF $f(x) \ni F'(x) = f(x)$
$\Rightarrow f(x)$ ASSUMES AS A VALUE EVERY NUMBER BETWEEN ANY 2 OF ITS VALUES. $f(1) = 1$, $f(1/\sqrt{2}) = 0$, BUT $\exists$ NO $x_0 \in [0,1] \ni f(x_0) = 1/2$, $\Rightarrow f(x)$ HAS NO PRIMITIVE

LET $\sigma(x) = K \ \forall \ x \in [0,1] \ni K \leq 0$
$T(x) = L \ \forall \ x \in [0,1] \ni L \geq 1 \Rightarrow \sigma(x) \leq f(x) \leq T(x)$
LET $\varepsilon = 1 \Rightarrow \int_0^1 [T(x) - \sigma(x)]dx \geq \varepsilon \Rightarrow f(x)$ IS NOT INTEGRABLE

11) ALTHOUGH $\int_a^x f(t)dt$ IS ALWAYS A PRIMITIVE OF $f(x)$, NOT EVERY PRIMITIVE OF $f(x)$ CAN BE WRITTEN $\int_a^x f(t)dt$

PROOF: $F(x) = \int_a^x f(t)dt \Rightarrow F'(x) = F'(x) - F'(a) = f(x)$
LET $f(x) = \cos x \ \ni F(x) = \sin x + 50$
$F(x) = \sin x + 50 = \int_a^x f(t)dt = F(x) - F(a)$, BUT $|F(a)| \leq 1$

12) SHOW THAT $f(x) = 0 \ \forall \ x \in [0,1] \ \& \ f(x) = 1 \ \forall \ x \in (1,2]$ IS INTEGRABLE ON $[0,2]$, THOUGH IT HAS NO PRIMITIVE THERE

PROOF: $f(x)$ IS A STEP FUNCTION $\Rightarrow f(x)$ IS INTEGRABLE
LET $F(x)$ BE A PRIMITIVE OF $f(x) \Rightarrow F'(x) = f(x) \Rightarrow f(x)$ ASSUMES AS A VALUE EVERY NUMBER BETWEEN ANY 2 OF ITS VALUES
$f(0) = 0$; $f(2) = 1$, BUT $\exists$ NO $x_0 \ni f(x_0) = 1/2 \Rightarrow \exists$ NO $F(x) \ni F'(x) = f(x)$

13) SHOW THAT $F(x) = x^2 \sin \frac{1}{x}$ $(F(0) = 0)$ HAS AN UNBOUNDED DERIVATIVE $f(x)$ ON $[0,1] \Rightarrow$ A FUNCTION MAY HAVE A PRIMATIVE WITHOUT BEING INTEGRABLE

$f(x) = F'(x) = 2x \sin 1/x^2 + \frac{1}{x} \cos 1/x^2$
$f(0^+) = -\infty \Rightarrow f(x)$ IS NOT BOUNDED $\Rightarrow f(x)$ IS NOT INTEGRABLE

14) $\int \tan^n x \, dx = \dfrac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$

$$\begin{aligned} \frac{d}{dx} \tan^p x &= p \sec^2 x \tan^{p-1} x \\ &= p(1 + \tan^2 x) \tan^{p-1} x \\ &= p \tan^{p-1} x + p \tan^{p+1} x \end{aligned}$$

$$\frac{\tan^p x}{p} = \int \tan^{p-1} x \, dx + \int \tan^{p+1} x \, dx$$

LET $n = p+1$

$$\frac{\tan^{n-1} x}{n-1} = \int \tan^{n-1} x \, dx + \int \tan^n x \, dx$$

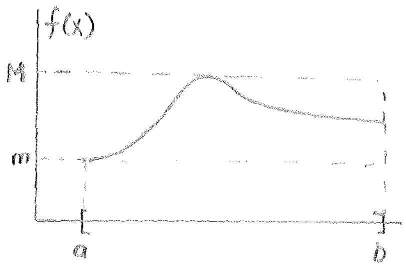
$$\Rightarrow \int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-1} x \, dx \quad \forall \ n \neq 1$$

Pg 117-8

5) IF $f(x)$ IS CONT. ON $[a,b]$ $\exists$ $\varepsilon \ni a<\varepsilon<b$

AND $\int_a^b f(x)dx = f(\varepsilon)\cdot(b-a)$

PROOF: $m = \text{MIN } f(x) = f(d)$ $\quad M = \text{MAX } f(x) = f(e)$



$$m(b-a) \le \int_a^b f(x)dx < M(b-a)$$

$$m \le \frac{\int_a^b f(x)dx}{b-a} \le M$$

A FUNCTION CONTINUOUS ON AN INTERVAL ASSUMES (AS A VALUE) EVERY NUMBER BETWEEN ANY TWO VALUES IT ASSUMES ON THAT INTERVAL

$$\therefore \exists\ f(\varepsilon) \in [m, M] \ni f(\varepsilon) = \frac{\int_a^b f(x)dx}{b-a}$$

$$\Rightarrow \int_a^b f(x)dx = f(\varepsilon)(b-a)$$

7) W.L.O.G., LET $b>a, \ni -b<-a<0<a<b$

$$\int_{-b}^b f(x)dx = 2\int_0^b f(x)dx$$

$$= 2\int_0^a f(x)dx + 2\int_a^b f(x)dx$$

$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$$

$$\Rightarrow \int_{-b}^b f(x)dx = \int_{-a}^a f(x)dx + 2\int_a^b f(x)dx$$

$$\int_{-b}^{-a} f(x)dx + \cancel{\int_{-a}^a f(x)dx} + \cancel{\int_a^b f(x)dx} = \cancel{\int_{-a}^a f(x)dx} + \cancel{2}\int_a^b f(x)dx$$

$$\Rightarrow \int_{-b}^{-a} f(x)dx = \int_a^b f(x)dx$$

9) $f(x) = f(-x) = -f(-x)$

$$f(x) = \frac{f(-x) - f(-x)}{2} = 0$$

10) $f(x) = \frac{1}{2}[f(x)+f(-x)] + \frac{1}{2}[f(x)-f(-x)]$

$= f_e(x) + f_o(x)$

$f_e(x) = \frac{1}{2}[f_e(x) + f_e(-x)]$

$2f_e(x) = f_e(x) + f_e(-x)$

$f_e(x) = f_e(-x)$

$\Rightarrow f_e(x)$ IS ODD

$f_o(x) = \frac{1}{2}[f_o(x) - f(-x)]$

$\Rightarrow f_o(x) = -f_o(-x)$

$\Rightarrow f_o(x)$ IS ODD

COURSE OBJECTIVES FOR MC_4

Material: Sec. 3.14-3.18, 4.12-4.16 of text.

Performance required for MC_4:

1. To write in words and appropriate symbols, any definition given in Sec. 3.14-3.18 or 4.12-4.16

2. To describe in words and appropriate symbols each of the possible types of indeterminant forms.

3. To write, using your own words, what is meant by the expression, "indeterminant form."

4. To write, in words and appropriate symbols, L'Hospital's Rule for any of the indeterminant forms discussed in Sec 3.14-3.18.

5. To write a Proof of L'Hospital's Rule, for one of Cases 1 - 4 p. 96.

6. To recognize an indeterminant form and to apply the appropriate version of L'Hospital's Rule to find the required limit. Sufficient examples should be provided in the text and in the exercise sec. 3.18 p.101.
   Note: Problems you will be required to work will NOT necessarily be from sec. 3.18, but WILL be similar.

To explain why integrals like

$$\int_0^1 dx/x \quad \text{and} \quad \int_a^{+\infty} e^{-x}\, dx \quad \text{are NOT covered by the}$$

definition of the integral given in sec. 4.01.

8. To identify, and explain why, a given improper integral is improper. There are 2 ways in which an integral may be improper.

9. To be able to state in words and prove the Comparison Test as given on p. 141.

10. To be able to use the definition of an improper integral, and, or the comparison theorem to determine convergence or divergence. In case of convergence, to be able to evaluate the integral, if required to do so. (Exercises 1-10, 15-25 of sec. 4.16.

11. To derive those values of the parameter α, for which

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x}\, dx \quad \text{(Gamma function) converges or diverges.}$$

12. Problems 28, 29, 30, p. 145.

To be able to explain why the Laplace Transform is an improper integral. In particular to explain why, for example

$$L\{e^{at}\} = \frac{1}{s-a} \quad , \text{for } s > a.$$

I) TO DESCRIBE EACH OF THE POSSIBLE INDETERMINATE FORMS

a) $\frac{0}{0} \Rightarrow \exists\ f(x) \text{ \& } g(x) \ni \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad [a = a, a^+, a^-, -\infty, +\infty, \infty]$

$\Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)}$ IS OF THE INDETERMINATE FORM $0/0$

b) $\frac{\infty}{\infty} \Rightarrow \exists\ f(x) \text{ \& } g(x) \ni \lim_{x\to a} |f(x)| = \lim_{x\to a} |g(x)| = \infty \quad [a = a, a^+, a^-, -\infty, +\infty, \infty]$

$\Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)}$ IS OF THE INDETERMINATE FORM $\frac{\infty}{\infty}$

c) $0\cdot\infty\ \exists\ f(x) \text{ \& } g(x) \ni \lim_{x\to a} f(x) = 0 \text{ \& } \lim_{x\to a} |g(x)] = \infty \quad [a = a, a^+, a^-, -\infty, +\infty, \infty]$

$\Rightarrow \lim_{x\to a} f(x)\,|g(x)|$ IS OF THE INDETERMINATE FORM $0\cdot\infty$

d) $\infty - \infty \Rightarrow \exists\ f(x) \text{ \& } g(x) \ni \lim_{x\to a} |f(x)| = \lim_{x\to a} |g(x)| = \infty \quad [a = a, a^+, a^-, -\infty, +\infty, \infty]$

$\Rightarrow \lim_{x\to a} [f(x) - g(x)]$ IS OF THE INDETERMINATE FORM $\infty - \infty$

e) $0^0 \Rightarrow \exists\ f(x) \text{ \& } g(x) \ni \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad [a = a, a^+, a^-, -\infty, +\infty, \infty]$

$\Rightarrow \lim_{x\to a} [f(x)]^{g(x)}$ IS OF THE INDETERMINATE FORM 0^0

f) $1^\infty \Rightarrow \exists\ f(x) \ni \lim_{x\to a} |f(x)| = \infty \quad [a = a, a^+, a^-, -\infty, +\infty, \infty]$

$\Rightarrow \lim_{x\to a} 1^{f(x)}$ IS OF THE INDETERMINATE FORM 1^∞

II) AN INDETERMINATE FORM IS AN EXPRESSION WHOSE VALUE EITHER FINITE OR INFINITE, AS X APROACHES A VALUE, CANNOT BE DETERMINED BY SUBSTITUTION OF THAT VALUE IN THE EXPRESSION

III) L'HOSPITAL'S RULES

1) IF $f(x)$ & $g(x)$ ARE DIFFERENTIABLE FUNCTIONS, AND $g'(x) \neq 0$ $\forall$ X IN A DELETED NEIGHBORHOOD OF ITS LIMIT, AND $\lim_{x\to D} f(x) = \lim_{x\to D} g(x) = 0$, $[D = a, a^+, a^-, -\infty, +\infty, \infty]$, AND IF $\lim_{x\to D} \frac{f'(x)}{g'(x)} = L$ $[L = \text{FINITE}, +\infty, -\infty, \infty]$, THEN $\lim_{x\to D} \frac{f(x)}{g(x)} = L$

2) IF $f(x)$ & $g(x)$ ARE DIFFERTIABLE FUNCTIONS, & $g'(x) \neq 0$ $\forall$ X IN A DELETED NEIGHBORHOOD OF ITS LIMIT, IF $\lim_{x\to D} f(x) = \lim_{x\to D} g(x) = \infty$ $[D = a, a^+, a^-, +\infty, -\infty, \infty]$ AND IF $\lim_{x\to D} \frac{f'(x)}{g'(x)} = L$ $[L = \text{FINITE}, +\infty, -\infty, \infty]$, THEN $\lim_{x\to D} \frac{f(x)}{g(x)} = L$

IV) IF $f(x)$ & $g(x)$ ARE DIFFERENTIABLE FUNCTIONS AND $g'(x) \neq 0$ $\forall$ X IN A DELETED NEIGHBORHOOD OF ITS LIMIT, AND $\lim f(x) = \lim g(x) = 0$, & $\lim \frac{f'(x)}{g'(x)} = L$, THEN $\lim \frac{f(x)}{g(x)} = L$

1) CASE 1: $x \to a^+$

LET $f(a) = g(a) = 0$

$f(x)$ & $g(x)$ ARE CONTINUOUS OVER A DELETED NEIGHBORHOOD $[a, a+\varepsilon] \ni \varepsilon > 0$. CHOOSE $\varepsilon \ni g'(x)$ DOES NOT VANISH

LET $x \in (a, a+\varepsilon] \Rightarrow \exists\, \xi \in (a, x) \ni$

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(x+\varepsilon) - f(x)}{g(x+\varepsilon) - g(x)}$$

$x \to a^+ \Rightarrow \xi \to a^+ \Rightarrow$

$$\lim_{\xi\to a^+} \frac{f'(\xi)}{g'(\xi)} = \lim_{x\to a^+} \frac{f(x+\varepsilon) - f(x)}{g(x+\varepsilon) - g(x)}$$

$$\Rightarrow \frac{f'(a)}{g'(a)} = \frac{f(a+\varepsilon)}{g(a+\varepsilon)}$$

AS $\varepsilon \to 0$; $\frac{f'(a)}{g'(a)} = \frac{f(a)}{g(a)}$

2) CASE 2: $x \to a^-$

LET $f(a) = g(a) = 0$

$f(x)$ ∮ $g(x)$ ARE CONTINUOUS OVER A DELETED NEIGHBORHOOD $[a-\varepsilon, a] \ni \varepsilon > 0$. CHOOSE $\varepsilon \ni g'(x)$ DOES NOT VANISH

LET $x \in [a-\varepsilon, a) \Rightarrow \exists\, \xi \in (a, x) \ni$

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(x) - f(x-\varepsilon)}{g(x) - g(x-\varepsilon)}$$

$$x \to a^- \Rightarrow \xi \Rightarrow a^- \Rightarrow \varepsilon \Rightarrow 0$$

$$\Rightarrow \lim_{\xi \to a^-} \frac{f'(\xi)}{g'(\xi)} = \lim_{x \to a^-} \frac{f(x) - f(x-\varepsilon)}{g(x) - g(x-\varepsilon)}$$

$$\xi \to a^- \Rightarrow f'(\xi) \to f'(a) \text{ ∮ } g'(\xi) \to g'(a)$$

$$x \to a^- \Rightarrow f(x) \to 0;\ g(x) \to 0;\ f(x-\varepsilon) \to f(a-\varepsilon) \to f(a)$$

ETC.

3) CASE 3: $x \to a$

LET $f(a) = g(a) = 0$

$f(x)$ ∮ $g(x)$ ARE CONTINUOUS OVER A DELETED NEIGHBORHOOD $[a-\varepsilon, a)(a, a+\varepsilon] \ni \varepsilon > 0$. CHOOSE $\varepsilon \ni g'(x)$ DOESN'T VANISH. SAME PROOFS AS CASES 1 ∮ 2

[$x \to a \Rightarrow x \to a^-$ AND $x \to a^+$].

4) CASE 4: $x \to +\infty$

$$L = \lim_{x \to +\infty} \frac{f'(x)}{g'(x)}$$

$$= \lim_{t \to 0^+} \frac{f'(1/t)}{g'(1/t)}$$

$$= \lim_{t \to 0^+} \frac{f'(1/t)\,t^{-2}}{g'(1/t)\,t^{-2}}$$

$$= \lim_{t \to 0^+} \frac{\frac{d}{dt} f(1/t)}{\frac{d}{dt} g(1/t)}$$

$$= \lim_{t \to 0^+} \frac{f(1/t)}{g(1/t)} = \lim_{x \to +\infty} \frac{f(x)}{g(x)}$$

V) EXCERCISES

1) $\lim_{x\to 2} \frac{3x^2+x-14}{x^2-x-2} \to 0 \Rightarrow \lim_{x\to 2}\frac{6x+1}{2x-1} = \frac{13}{3}$

2) $\lim_{x\to -3}\frac{x^3+x+30}{4x^3+11x^2+9}\to 0 \Rightarrow \lim_{x\to -3}\frac{3x^2+1}{12x^2+22} = \frac{28}{130}=\frac{14}{65}$

3) $\lim_{x\to 1}\frac{\ln x}{x^2+x-2}\to 0 \Rightarrow \lim_{x\to 1}\frac{\frac{1}{x}}{2x+1} = \frac{1}{3}$

4) $\lim_{x\to 1}\frac{\cos\frac{\pi}{2}x}{x-1}\to 0\Rightarrow \frac{-\frac{\pi}{2}\sin\frac{\pi}{2}x}{1} = -\frac{\pi}{2}$

5) $\lim_{x\to 0}\frac{\cos x-1+\frac{1}{2}x^2}{x^4}\to 0\Rightarrow\lim_{x\to 0}\frac{-\sin x+x}{4x^3}\to 0\Rightarrow\lim_{x\to 0}\frac{-\cos x+1}{12x^2}\Rightarrow\lim_{x\to 0}\frac{\sin x}{24x}\Rightarrow\lim_{x\to 0}\frac{\cos x}{24}=\frac{1}{24}$

6) $\lim_{x\to 0}\frac{\ln(1+x)-x}{\cos x-1}\to 0\Rightarrow\lim_{x\to 0}\frac{\frac{1}{1+x}-1}{-\sin x}=\lim_{x\to 0}\frac{1-x+1}{-(x+1)\sin x}=\lim_{x\to 0}\frac{+x}{+(x+1)\sin x}\to 0\Rightarrow$

$\lim_{x\to 0}\frac{1}{(x+1)\cos x+\sin x}=1$

8) $\lim_{x\to\pi}\frac{\sin^2 x}{\tan^2 4x}=\lim_{x\to\pi}\frac{\sin^2 x\cos^2 4x}{\sin^2 4x}$ 1

9) $\lim_{x\to 0}\frac{a^x-1}{b^x-1}\to 0\Rightarrow\lim_{x\to 0}\frac{x a^{x-1}}{x b^{x-1}}=\lim_{x\to 0}\frac{a^{x-1}}{b^{x-1}}=\frac{b}{a}$

11) $\lim_{x\to\infty}\frac{8x^5-5x^2+1}{3x^5+1}\Rightarrow\lim_{x\to\infty}\frac{40x^4-10x}{15x^4}\Rightarrow\lim_{x\to\infty}\frac{160x^3-10}{60x^3}=\frac{16}{6}=\frac{8}{3}$

12) $\lim_{x=\pi/2}\frac{\tan x-6}{\sec x+5}\Rightarrow\lim_{x\to\pi/2}\frac{\sec^2 x}{\sec x\tan x}=\lim_{x\to\pi/2}\frac{\sec x}{\tan x}=\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{\cos x\sin x}=1$

13) $\lim_{x=\frac{\pi}{2}}\frac{\ln\sin 2x}{\ln\cos x}\Rightarrow\lim_{x\to\frac{\pi}{2}}\frac{\frac{2\cos 2x}{\sin 2x}}{\frac{-\sin x}{\cos x}}=\lim_{x\to\frac{\pi}{2}}\frac{2\cos 2x\cos x}{-\sin 2x\sin x}\Rightarrow\lim_{x\to\frac{\pi}{2}}\frac{-2}{\tan 2x\tan x}=1$

19) $\lim_{x\to 0^+}x(\ln x)^n\Rightarrow 0\cdot-\infty^n\Rightarrow\lim_{x\to 0^+}\frac{(\ln x)^n}{x^{-1}}\Rightarrow\lim_{x\to 0^+}\frac{-n(\ln x)^{n-1}}{x\cdot x^{-2}}=$

24) $y=(1+2\sin x)^{\cot x}\Rightarrow\ln y=\cot x\ln(1+2\sin x)$

$\lim_{x\to 0}\frac{\ln(1+2\sin x)}{\tan x}\Rightarrow\lim_{x\to 0}\frac{2\cos x}{(1+2\sin x)\sec^2 x}=\lim_{x\to 0}\frac{2\cos^3 x}{(1+2\sin x)}=\frac{2}{1}$

$\lim_{x\to 0}\ln y=2\Rightarrow\lim_{x\to 0}y=e^2$

27) $\lim_{x\to 0}x^{x^x}$ $y=x^{x^x}\Rightarrow\ln y=x^x\ln x\Rightarrow\ln\ln y=x\ln x+\ln\ln x$

$\lim_{x\to 0}(x\ln x+\ln\ln x)=0\cdot\infty+-\infty$ ARG!

$\ln + \frac{x}{x} + \frac{1}{x\ln x}$

$\lim_{x\to 0}\left[\ln x+1\right] + \lim\left(\frac{1}{x^2} + \frac{\ln x}{x^2}\right)$

$\frac{1}{x}$

VI) DEFINITIONS

1) LET $f(x)$ BE RIEMAN INTEGRABLE ON $[a, b-\varepsilon]$ ($[a+\varepsilon, b)$) $\forall\ \varepsilon \ni$ $0<\varepsilon<b-a$, BUT NOT INTEGRABLE ON $[a,b]$ AND ASSUME THAT $\lim_{\varepsilon\to 0^+}\int_a^{b-\varepsilon} f(x)dx$ ($\lim_{\varepsilon\to 0^+}\int_{a+\varepsilon}^{b} f(x)dx$) EXISTS. THE INPROPER INTEGRAL $\int_a^b f(x)dx$ IS THEN DEFINED AS:

$$\int_a^b f(x)dx \equiv \lim_{\varepsilon\to 0^+}\int_a^{b-\varepsilon} f(x)dx \quad \left(\lim_{\varepsilon\to 0^+}\int_{a+\varepsilon}^{b} f(x)dx\right)$$

IF THIS LIMIT IS FINITE, THE INTEGRAL $\int_a^b f(x)dx$ IS CONVERGENT TO THIS LIMIT, AND $f(x)$ IS SAID TO BE INPROPERLY INTEGRABLE ON $[a,b)$ ($(a,b]$)

IF THIS LIMIT IS INFINITE, OR DOESN'T EXIST, THE INTEGRAL IS DIVERGENT

2) LET $[a,b]$ BE FINITE; $c\in(a,b)$, AND LET $\int_a^c f(x)dx$ & $\int_c^b f(x)dx$ BE CONVERGENT INPROPER INTERVALS. THEN $\int_a^b f(x)dx$ IS CONVERGENT: $\int_a^b f(x)dx \equiv \int_a^c f(x)dx + \int_c^b f(x)dx$

IF EITHER OF THE TWO RT. HAND INT. DIVERGE, SO DOES $\int_a^b f(x)dx$

3) LET $f(x)$ BE RIEMAN-INTEGRABLE ON $[a,U]$ $\forall\ U>a$ & ASSUME $\lim_{U\to+\infty} f(x)dx$ EXISTS. THE IMPROPER INTEGRAL

$$\int_a^{+\infty} f(x)dx \equiv \lim_{U\to+\infty}\int_a^U f(x)dx$$

IF $\int_a^{+\infty} f(x)dx$ IS FINITE, THE IMPROPER INTEGRAL IS CONVERGENT TO THIS LIMIT, AN $f(x)$ IS IMPROPERLY INTEGRABLE ON $[a,+\infty)$

IF $\int_a^{+\infty} f(x)dx$ IS INFINITE, OR DOESN'T EXIST, THEN THE INTEGRAL IS DIVERGENT

4) LET $f(x)$ BE RIEMAN-INTEGRABLE ON EVERY $[a,b]$, AND ASSUME IMPROPER INTEGRALS $\int_0^{+\infty} f(x)dx$ & $\int_{-\infty}^{0} f(x)dx$ CONVERGE. THEN THE IMPROPER INTEGRAL

$$\int_{-\infty}^{\infty} f(x)dx \equiv \int_{-\infty}^{0} f(x)dx + \int_0^{\infty} f(x)dx$$

IF EITHER RT. HAND LIMIT DIVERGES, SO DOES $\int_{-\infty}^{\infty} f(x)dx$

5) LET $f(x)$ BE INPROPERLY INTEGRABLE ON $(a,c]$ & $[c,+\infty)$ $\ni c>a$. THEN THE INPROPER INTEGRABLE

$$\int_a^{+\infty} f(x)dx \equiv \int_a^c f(x)dx + \int_c^{+\infty} f(x)dx \quad \text{IS CONVERGENT.}$$

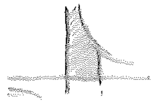
IF EITHER RT. HAND LIMIT DIVERGES, SO DOES $\int_a^{+\infty} f(x)dx$

6) IF $g(x)$ DOMINATES $f(x)$ ON A SET $A \Rightarrow g(x)$ & $f(x)$ ARE DEFINED $\forall x\in A$, AND $|f(x)|\leq g(x)$ $\forall\ x$

7)

II) EXPLAIN WHY THE FOLLOWING FUNCTIONS ARE NOT COVERED BY DEF. $\int_a^b f(x)dx \equiv \lim_{|n| \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$

1) $\int_0^1 \frac{dx}{x}$



$\lim_{|n| \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$ DOES NOT EXIST FOR UNBOUNDED FUNCTIONS $[f(0) = +\infty]$

FOR x_i CAN BE CHOSEN $\ni \sum_{i=1}^{n} f(x_i)\Delta x_i$ IS LARGER THAN ANY OTHER PREASSIGNED QUANTITY

2) $\int_a^{+\infty} e^{-x}dx$: DEFINITION ONLY DEALS WITH FINITE INTERVALS; $[a, b]$

III) THE TWO WAYS AN INTEGRAL IS IMPROPER

1) $\int_a^b f(x)dx$ IS IMPROPER IF $f(x)$ IS NOT BOUNDED ON $[a,b]$

2) AN INTEGRAL IS IMPROPER IF EVALUATED OVER AN INFINITE INTERVAL

IV) THE COMPARISON TEST: IF $g(x)$ DOMINATES not necessary NON-NEGATIVE $f(x)$ ie $g(x) \geq |f(x)|$ ON $[a, b)$ $([a, +\infty))$ AND BOTH ARE INTEGRABLE ON $[a, c]$ $\forall c \in (a, b)$ $(a < c)$, AND IF $\int_a^b g(x)dx$ $\left[\int_a^{+\infty} g(x)dx\right]$ CONVERGES, THEN SO DOES $\int_a^b f(x)dx$ $\left[\int_a^{+\infty} f(x)dx\right]$

PROOF: $g(x) \geq 0$ & $f(x) \geq 0 \Rightarrow$ BOTH $\int_a^c f(x)dx$ & $\int_a^c g(x)dx$ ARE BOTH MONOTONICALLY INCREASING FUNCTIONS OF C, & BOTH HAVE LIMITS AS $c \to b^-$ $(c \to +\infty)$

$0 \leq \int_a^c f(x)dx \leq \int_a^c g(x)dx \Rightarrow 0 \leq \int_a^b f(x)dx \leq \int_a^b g(x)dx$

$(f(x) \leq g(x)$ AND $\lim f(x)$ & $\lim g(x)$ EXIST $\Rightarrow \lim f(x) \leq \lim g(x))$

$\left(0 \leq \int_a^{+\infty} f(x)dx \leq \int_a^{+\infty} g(x)\right)$ Q.E.D.

f(x) dominates g(x) on a set A
iff both functions are defined $\forall \in A$ and $\forall x$ $|g(x)| \leq f(x)$

V) EXERCISES

1) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

$x = 3\sin\theta \Rightarrow dx = 3\cos\theta\, d\theta$

$\int \frac{3\cos\theta}{3\cos\theta} d\theta = \sin^{-1}\frac{x}{3}\Big]_0^3$

$= \frac{\pi}{2}$

2) $\int_{-1}^{1} x^{-\frac{1}{3}} dx = \int_{-1}^{0} x^{-\frac{1}{3}} dx + \int_0^1 x^{-\frac{1}{3}} dx$

$= \frac{-3}{4x^{\frac{4}{3}}}\Big]_{-1}^{-\varepsilon} + \frac{-3}{4x^{4/3}}\Big]_{\varepsilon}^{+1}$

$= \frac{-3}{4(-\varepsilon)^{\frac{4}{3}}} - \frac{3}{4(-1)^{4/3}} + \frac{-3}{4(1)} + \frac{3}{4\varepsilon^{4/3}}$

$= 0$

3) $\int_{-2}^{2} x^{-3} dx = \int_{-2}^{-\varepsilon} x^{-3} dx + \int_{+\varepsilon}^{2} x^{-3} dx$

$\Rightarrow -\frac{1}{4x^4}\Big]_{-2}^{-\varepsilon} - \frac{1}{4x^4}\Big]_{+\varepsilon}^{2}$

$= \frac{-1}{4\varepsilon^4} + \frac{1}{64} - \frac{1}{4\varepsilon^4} + \frac{1}{64}$

$\Rightarrow \lim_{\varepsilon \to 0} \frac{-2}{4\varepsilon^4} = \infty \Rightarrow$

$\Rightarrow \int_{-2}^{2} x^{-3} dx$ IS DIVERGENT

4) $\int_0^{\pi/2} (\sin x \tan x)^{\frac{1}{2}} dx$

$\int_0^{\pi/2} \frac{\sin x}{\sqrt{\cos x}} dx$ ARG!

2

5) $\int_0^{+\infty} e^{-\frac{x}{2}} dx$

$= -2e^{-\frac{x}{2}}\Big]_0^{+\infty} = 2$

6) $\int_0^{+\infty} \sin x\, dx$

$= -\cos x\Big]_0^{+\infty}$

$-\cos\infty$ CAN'T BE DETERMINED

$\Rightarrow \int_0^{+\infty} \sin x\, dx$ IS INDETERMINATE

7) $\int_1^{+\infty} \frac{dx}{x\sqrt{x^2-1}} dx$

$x = \sec\theta \Rightarrow dx = \sec\theta\tan\theta\, d\theta$

$\int \frac{\sec\theta\tan\theta}{\sec\theta\tan\theta} d\theta$

$\Rightarrow [\sec^{-1} x]_1^{+\infty}$

$= \frac{\pi}{2}$

9) $\int_0^{+\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$

$= 2e^{-\sqrt{x}}\Big]_0^{+\infty} = 2$

10) $\int_{-\infty}^{\infty} \frac{dx}{1+4x^2}$

$x = \frac{1}{2}\tan\theta$

$\Rightarrow dx = \frac{1}{2}\sec^2\theta\, d\theta$

$\int \frac{\frac{1}{2}\sec^2\theta\, d\theta}{\sec^2\theta} = \int \frac{d\theta}{2}$

$\Rightarrow \frac{1}{2}[\tan^{-1} 2x]_{-\infty}^{\infty}$

$= \frac{1}{2}\left[\frac{3\pi}{2} - \frac{\pi}{2}\right] = \frac{\pi}{2}$

VI) DERIVE THE VALUES $\alpha \ni$

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx \quad \text{CONVERGES OR DIVERGES}$$

DERIVATION:

$$\int_0^{+\infty} x^{\alpha-1} e^{-x} dx = \int_0^1 x^{\alpha-1} e^{-x} dx + \int_1^{+\infty} x^{\alpha-1} e^{-x}$$

LET $f(x) = x^{\alpha-1} e^{-x} dx$

$g_0(x) = x^{\alpha-1}$

$g_\infty(x) = 1/x^2$

1) $\lim_{x \to 0^+} \frac{f(x)}{g_0(x)} = \lim_{x \to 0^+} e^{-x} = 1 ; \; 0 < 1 < +\infty$

x IS OF THE SAME ORDER AS $x \to 0^+$

$\Rightarrow x^{\alpha-1}$ AND $x^{\alpha-1} e^{-x}$ ARE OF THE SAME ORDER AS $x \to 0^+$

$\int_0^1 x^{\alpha-1} dx$ CONVERGES IFF $\alpha > 0$

2) $\lim_{x \to +\infty} \frac{f(x)}{g_\infty(x)} = \lim_{x \to +\infty} x^2 [x^{\alpha-1} e^{-x}] = \lim_{x \to +\infty} \frac{x^{\alpha+1}}{e^x} = 0; \; 0 \leq L < +\infty$

$\Rightarrow f(x) = O[g_\infty(x)]$ AS $x \to +\infty$

$\int_1^{+\infty} \frac{dx}{x^2}$ CONVERGES $\forall \alpha \Rightarrow \int_1^{+\infty} x^{\alpha-1} e^{-x} dx$ CONVERGES $\forall \alpha$

$\Rightarrow \int_0^{+\infty} x^{\alpha-1} e^{-x} dx$ IS CONVERGENT $\forall \alpha > 0$, AND IS DIVERGENT OTHERWISE

VII) PROVE:

$$\int_a^{+\infty} \frac{dx}{x \cdot \ln x \cdot \ln \ln x \cdot \ldots \cdot (\ln \ln \ldots \ln x)^p} \quad \text{CONVERGES}, \; a > 0$$

IFF $p > 1$

PROOF: let $f(x) = \frac{1}{x \cdot \ln x \cdot \ln \ln x \cdot \ldots \cdot (\ln \ln \ldots \ln x)^p}$

$g(x) = x^{1-p}$

$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{x^{-p}}{\ln x \cdot \ln \ln x \cdot \ldots \cdot (\ln \ln \ldots \ln x)^p} = 0$ IF $p > 0$

$\Rightarrow f(x) = O[g(x)] \Rightarrow f(x) \leq K g(x)$

$\int_a^{+\infty} x^{1-p} dx = (1-p) x^{-p} \Big]_a^{+\infty}$ WHICH CONVERGES $\forall p > 1$